MATH332-Linear Algebra	**Homework Four Solutions**

Matrix Inversion, Decomposition and Determinants

Text: 2.2-2.5, 3.1-3.3 Section Overviews: 2.2-2.5, 3.1-3.3

Quote of Homework Four Solutions	
And my tears in league with the wires and energy and my machine.	
	Underworld : Cowgirl (1994)

1. Matrix Inversion

Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

1.1. **Matrix Inverse: Take One.** Find $\mathbf{A}^{-1}$ using the Gauss-Jordan method.[1]
When asked to calculate an inverse matrix this is the algorithm to use. It is simpler and less computationally intensive than other methods and is roughly what a computational device does when asked to find an inverse matrix.

$$\left[\begin{array}{ccc|ccc} 3 & 6 & 7 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R1 \to R1 - 3R2 \\ \sim \\ R3 \to 2R1 - 3R3 \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 4 & 1 & -3 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 2 & 0 & -3 \end{array}\right] \begin{array}{c} \\ \sim \\ 2R3 - 3R2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 3 & 0 & 4 & 1 & -3 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right] \begin{array}{c} R1 \to R1 - 4R3 \\ \sim \\ R2 \to R2 - R3 \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & -15 & 9 & 24 \\ 0 & 2 & 0 & -4 & 4 & 6 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right] \begin{array}{c} R1 \to R1/3 \\ \sim \\ R2 \to R2/2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & 8 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right] \implies \quad A^{-1} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$$

1.2. **Matrix Inverse: Take Two.** Find $\mathbf{A}^{-1}$ using the cofactor representation. [2]
There are, of course, other ways to find $\mathbf{A}^{-1}$. The following method uses determinants and provides a general representation of an inverse matrix, if it exists. First we must find $\det(\mathbf{A})$. Using the cofactor expansion of the determinant we have,

$$\begin{aligned} \det(\mathbf{A}) &= 3\det\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} - 0 \cdot \det\begin{pmatrix} 6 & 7 \\ 3 & 4 \end{pmatrix} + 2\det\begin{pmatrix} 6 & 7 \\ 2 & 1 \end{pmatrix} \\ &= 3(5) - 0(3) + 2(-8) = 15 - 16 = -1 \end{aligned}$$

Using the cofactor formula we have,

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} 5 & -3 & -8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix},$$

where $c_{ij} = (-1)^{i+j}\det(\mathbf{A}_{ij})$. Since, this method requires the use of determinants it is computationally intensive, but does highlight the connection between $\det(\mathbf{A}) = 0$ and non-invertibility. A typical use of this method is to study how elements of $\mathbf{A}^{-1}$ changes with changes to $\mathbf{A}$.

[1] The Gauss-Jordan method is another name for row-reduction. For an example see page 124 of the text.
[2] Though row-reduction is more efficient, it is sometimes that case that the whole inverse isn't needed. If particular entries of the inverse matrix are needed then one can use the general inversion formula given by theorem 8 on page 203, which consists of a matrix populated by cofactors.

1.3. **Check Step.** Verify that this inverse matrix is correct.

It is easy to verify that we have found the correct matrix inverse of $\mathbf{A}$. We have found $\mathbf{A}^{-1}$ using two different methods and gotten the same answer but if we are still worried then we conduct the following matrix multiplication $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. Doing so gives,

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I},$$

which implies that $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$.

1.4. **Solutions to Linear Systems.** Using $\mathbf{A}^{-1}$ find the unique solution to $\mathbf{A}\mathbf{x} = \mathbf{b} = [b_1\, b_2\, b_3]^{\mathsf{T}}$.

Since there is an inverse matrix for $\mathbf{A}$ there must exist a unique solution regardless of the choice of $\mathbf{b} \in \mathbb{R}^3$. Algebraically we have, $\mathbf{A}\mathbf{x} = \mathbf{b} \iff \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, where

$$\begin{aligned} \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\ &= \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ &= \begin{bmatrix} -5b_1 + 3b_2 + 8b_3 \\ -2b_1 + 2b_2 + 3b_3 \\ 4b_1 - 3b_2 - 6b_3 \end{bmatrix}. \end{aligned}$$

1.5. **Left Inversion in Rectangular Cases.** Let $\mathbf{A}^{-1}_{\text{left}} = (\mathbf{A}^{\mathsf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{T}}$. Show that $\mathbf{A}^{-1}_{\text{left}}\mathbf{A} = \mathbf{I}$. [3]
Assuming that the matrix is of full rank, this formula will work whether or not the matrix $\mathbf{A}$ is square.[4] If they are square then you can use properties of inverse matrices to show that $\mathbf{A}^{-1}_{\text{left}} = \mathbf{A}^{-1}$. Generally, we have,

$$\mathbf{A}^{-1}_{\text{left}}\mathbf{A} = (\mathbf{A}^{\mathsf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{T}}\mathbf{A} = \mathbf{I}. \tag{1}$$

The importance is on the dimensions of the matrices involved. If $\mathbf{A} \in \mathbb{R}^{m\times n}$ then $\mathbf{A}^{\mathsf{T}}\mathbf{A} \in \mathbb{R}^{n\times n}$, which implies that the identity matrix here is $\mathbf{I}_{n\times n}$.

1.6. **Right Inversion in Rectangular Cases.** Let $\mathbf{A}^{-1}_{\text{right}} = \mathbf{A}^{\mathsf{T}}(\mathbf{A}\mathbf{A}^{\mathsf{T}})^{-1}$. Show that $\mathbf{A}\mathbf{A}^{-1}_{\text{right}} = \mathbf{I}$.[5]
Assuming that the matrix is of full rank, this formula will work whether or not the matrix $\mathbf{A}$ is square.[6] If they are square then you can use properties of inverse matrices to show that $\mathbf{A}^{-1}_{\text{right}} = \mathbf{A}^{-1}$. Generally, we have,

$$\mathbf{A}\mathbf{A}^{-1}_{\text{right}} = \mathbf{A}\mathbf{A}^{\mathsf{T}}(\mathbf{A}\mathbf{A}^{\mathsf{T}})^{-1} = \mathbf{I}. \tag{2}$$

The importance is on the dimensions of the matrices involved. If $\mathbf{A} \in \mathbb{R}^{m\times n}$ then $\mathbf{A}\mathbf{A}^{\mathsf{T}} \in \mathbb{R}^{m\times m}$, which implies that the identity matrix here is $\mathbf{I}_{m\times m}$.

1.7. **Inversion for Rectangular Matrices.** Let $\mathbf{A}_1 = [2\ 2]^{\mathsf{T}}$ and $\mathbf{A}_2 = [2\ 2]$. Using the previous formula find the left-inverse of $\mathbf{A}_1$ and the right-inverse of $\mathbf{A}_2$. Check your results with the appropriate multiplication.
This is just a matter of checking the formula. We already know that for $\mathbf{A}_1$ we should get a one-by-one identity matrix. Letting $\mathbf{A}_1 = \mathbf{A}$ for the following steps, we get,

[3] This matrix is called the left-inverse of $\mathbf{A}$ and it can be shown that if $\mathbf{A} \in \mathbb{R}^{m\times n}$ such that $\mathbf{A}$ has a pivot in every column then the left inverse exists.

[4] A matrix is of full rank if it has as many pivots as possible.

[5] This matrix is called the right-inverse of $\mathbf{A}$ and it can be shown that if $\mathbf{A} \in \mathbb{R}^{m\times n}$ such that $\mathbf{A}$ has a pivot in every row then the right inverse exists.

[6] A matrix is of full rank if it has as many pivots as possible.

$$\mathbf{A}_{\text{left}}^{-1}\mathbf{A} = (\mathbf{A}^{\mathsf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{T}}\mathbf{A} = \left(\begin{bmatrix} 2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right)^{-1}\begin{bmatrix} 2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{3}$$

$$= \frac{1}{8}\begin{bmatrix} 2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{4}$$

$$= \frac{1}{8}8 = 1. \tag{5}$$

For the right inverse we let $\mathbf{A}_2 = \mathbf{A}$ for the following steps gives,

$$\mathbf{A}\mathbf{A}_{\text{right}}^{-1} = \mathbf{A}\mathbf{A}^{\mathsf{T}}(\mathbf{A}\mathbf{A}^{\mathsf{T}})^{-1} \tag{6}$$

$$= \begin{bmatrix} 2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix}\left(\begin{bmatrix} 2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right)^{-1} \tag{7}$$

$$= 1. \tag{8}$$

2. Block Matrix Inversion

Suppose that $\mathbf{A} \in \mathbb{R}^{n\times n}$ can be written in partitioned form as,

$$\mathbf{A} = \begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{bmatrix}. \tag{9}$$

2.1. Inverse Formula One.

Suppose that $\mathbf{A}$ and $\mathbf{P}$ are non-singular and show that,

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{X} & -\mathbf{P}^{-1}\mathbf{Q}\mathbf{W} \\ -\mathbf{W}\mathbf{R}\mathbf{P}^{-1} & \mathbf{W} \end{bmatrix}, \tag{10}$$

where $\mathbf{W} = (\mathbf{S} - \mathbf{R}\mathbf{P}^{-1}\mathbf{Q})^{-1}$ and $\mathbf{X} = \mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{Q}\mathbf{W}\mathbf{R}\mathbf{P}^{-1}$.[7]
We consider the following block multiplication,

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{P}\mathbf{X} - \mathbf{Q}\mathbf{W}\mathbf{R}\mathbf{P}^{-1} & -\mathbf{Q}\mathbf{W} + \mathbf{Q}\mathbf{W} \\ \mathbf{R}\mathbf{X} - \mathbf{S}\mathbf{W}\mathbf{R}\mathbf{P}^{-1} & -\mathbf{R}\mathbf{P}^{-1}\mathbf{Q}\mathbf{W} + \mathbf{S}\mathbf{W} \end{bmatrix}. \tag{11}$$

Working out the blocks gives,

$$\left[\mathbf{A}\mathbf{A}^{-1}\right]_{11} = \mathbf{P}\mathbf{X} - \mathbf{Q}\mathbf{W}\mathbf{R}\mathbf{P}^{-1} \tag{12}$$

$$= \mathbf{P}\left(\mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{Q}\mathbf{W}\mathbf{R}\mathbf{P}^{-1}\right) - \mathbf{Q}\mathbf{W}\mathbf{R}\mathbf{P}^{-1} \tag{13}$$

$$= \mathbf{I}, \tag{14}$$

$$\left[\mathbf{A}\mathbf{A}^{-1}\right]_{21} = \mathbf{R}\mathbf{X} - \mathbf{S}\mathbf{W}\mathbf{R}\mathbf{P}^{-1} \tag{15}$$

$$= \mathbf{R}\left(\mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{Q}\mathbf{W}\mathbf{R}\mathbf{P}^{-1}\right) - \left(\mathbf{W}^{-1} + \mathbf{R}\mathbf{P}^{-1}\mathbf{Q}\right)\mathbf{W}\mathbf{R}\mathbf{P}^{-1} \tag{16}$$

$$= \mathbf{R}\mathbf{P}^{-1} + \mathbf{R}\mathbf{P}^{-1}\mathbf{Q}\mathbf{W}\mathbf{R}\mathbf{P}^{-1} - \mathbf{R}\mathbf{P}^{-1} - \mathbf{R}\mathbf{P}^{-1}\mathbf{Q}\mathbf{W}\mathbf{R}\mathbf{P}^{-1} \tag{17}$$

$$\left[\mathbf{A}\mathbf{A}^{-1}\right]_{22} = -\mathbf{R}\mathbf{P}^{-1}\mathbf{Q}\mathbf{W} + \mathbf{S}\mathbf{W} \tag{18}$$

$$= -\mathbf{R}\mathbf{P}^{-1}\mathbf{Q}\mathbf{W} + \left(\mathbf{W}^{-1} + \mathbf{R}\mathbf{P}^{-1}\mathbf{Q}\right)\mathbf{W} \tag{19}$$

$$= -\mathbf{R}\mathbf{P}^{-1}\mathbf{Q}\mathbf{W} + \mathbf{I} + \mathbf{R}\mathbf{P}^{-1}\mathbf{Q}\mathbf{W} \tag{20}$$

$$\mathbf{I} \tag{21}$$

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[7]Hint: First, remember that if you are given a candidate for an inverse then you need only check that the appropriate multiplication gives you the identity. Second, you must note that we are working with a matrix whose elements are matrices and when you perform a check you are checking blocks. Thus, when you perform the check $\left[\mathbf{A}\mathbf{A}^{-1}\right]_{11}$ you are finding the upper-left block of the product matrix and the result should be matrix and not a scalar. What matrix should you get for this block? What about the rest?

2.2. **Inversion Formula Two.** Suppose that $\mathbf{A}$ and $\mathbf{S}$ are non-singular and show that,

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{X} & -\mathbf{X}\mathbf{Q}\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\mathbf{R}\mathbf{X} & \mathbf{W} \end{bmatrix}, \tag{22}$$

where $\mathbf{X} = (\mathbf{P} - \mathbf{Q}\mathbf{S}^{-1}\mathbf{R})^{-1}$ and $\mathbf{W} = \mathbf{S}^{-1} + \mathbf{S}^{-1}\mathbf{R}\mathbf{X}\mathbf{Q}\mathbf{S}^{-1}$.[8]
We now consider the following multiplication,

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} \mathbf{X}\mathbf{P} - \mathbf{X}\mathbf{Q}\mathbf{S}^{-1}\mathbf{R} & \mathbf{X}\mathbf{Q} - \mathbf{X}\mathbf{Q} \\ -\mathbf{S}^{-1}\mathbf{R}\mathbf{X}\mathbf{P} + \mathbf{W}\mathbf{R} & -\mathbf{S}^{-1}\mathbf{R}\mathbf{X}\mathbf{Q} + \mathbf{W}\mathbf{S} \end{bmatrix}, \tag{23}$$

via the blocks,

$$\begin{aligned}
\left[\mathbf{A}^{-1}\mathbf{A}\right]_{11} &= \mathbf{X}\mathbf{P} - \mathbf{X}\mathbf{Q}\mathbf{S}^{-1}\mathbf{R} && (24)\\
&\mathbf{X}\left(\mathbf{X}^{-1} + \mathbf{Q}\mathbf{S}^{-1}\mathbf{R}\right) - \mathbf{X}\mathbf{Q}\mathbf{S}^{-1}\mathbf{R} && (25)\\
&= \mathbf{I} + \mathbf{X}\mathbf{Q}\mathbf{S}^{-1}\mathbf{R} - \mathbf{X}\mathbf{Q}\mathbf{S}^{-1}\mathbf{R} && (26)\\
&= \mathbf{I}, && (27)
\end{aligned}$$

$$\begin{aligned}
\left[\mathbf{A}^{-1}\mathbf{A}\right]_{21} &= -\mathbf{S}^{-1}\mathbf{R}\mathbf{X}\mathbf{P} + \mathbf{W}\mathbf{R} && (28)\\
&= -\mathbf{S}^{-1}\mathbf{R}\left(\mathbf{X}^{-1} + \mathbf{Q}\mathbf{S}^{-1}\mathbf{R}\right) + \left(\mathbf{S}^{-1} + \mathbf{S}^{-1}\mathbf{R}\mathbf{X}\mathbf{Q}\mathbf{S}^{-1}\right)\mathbf{R} && (29)\\
&= -\mathbf{S}^{-1}\mathbf{R} - \mathbf{S}^{-1}\mathbf{R}\mathbf{Q}\mathbf{S}^{-1}\mathbf{R} + \mathbf{S}^{-1}\mathbf{R} + \mathbf{S}^{-1}\mathbf{R}\mathbf{X}\mathbf{Q}\mathbf{S}^{-1}\mathbf{R} = && (30)\\
&= \mathbf{0}, && (31)
\end{aligned}$$

$$\begin{aligned}
\left[\mathbf{A}^{-1}\mathbf{A}\right]_{22} &= -\mathbf{S}^{-1}\mathbf{R}\mathbf{X}\mathbf{Q} + \mathbf{W}\mathbf{S} && (32)\\
&= -\mathbf{S}^{-1}\mathbf{R}\mathbf{X}\mathbf{Q} + \left(\mathbf{S}^{-1} + \mathbf{S}^{-1}\mathbf{R}\mathbf{X}\mathbf{Q}\mathbf{S}^{-1}\right)\mathbf{S} && (33)\\
&= -\mathbf{S}^{-1}\mathbf{R}\mathbf{X}\mathbf{Q} + \mathbf{I} + \mathbf{S}^{-1}\mathbf{R}\mathbf{X}\mathbf{Q} && (34)\\
&= \mathbf{I}. && (35)
\end{aligned}$$

2.3. **Conclusion.** Show that if $\mathbf{P}, \mathbf{S}, \mathbf{A}$ are all non-singular matrices then $(\mathbf{S} - \mathbf{R}\mathbf{P}^{-1}\mathbf{Q})^{-1} = \mathbf{S}^{-1} + \mathbf{S}^{-1}\mathbf{R}\mathbf{X}\mathbf{Q}\mathbf{S}^{-1}$.
We have the existence of an inverse matrix. It is known that if an inverse matrix exists then it is unique. Consequently, the inverse from part 2 must be equal to the inverse in part 1. Comparing their lower left blocks gives that $(\mathbf{S} - \mathbf{R}\mathbf{P}^{-1}\mathbf{Q})^{-1} = \mathbf{S}^{-1} + \mathbf{S}^{-1}\mathbf{R}\mathbf{X}\mathbf{Q}\mathbf{S}^{-1}$.

2.4. **Sanity Check.** Test these formula with $\mathbf{P} = a$, $\mathbf{Q} = b$, $\mathbf{R} = c$, $\mathbf{S} = d$, where $a, b, c, d \in \mathbb{R}$ such that $ad - cb \neq 0$.

Assuming that scalar blocks with nonzero determinant gives,

$$\mathbf{X} = \mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{Q}\mathbf{W}\mathbf{R}\mathbf{P}^{-1} = \frac{1}{a} + \frac{b}{a}\mathbf{W}\frac{c}{a}, \tag{36}$$

where $\mathbf{W} = a/(ad - bc)$. Thus, $\mathbf{X} = d/(ad - bc)$. Also,

$$\begin{aligned}
-\mathbf{P}^{-1}\mathbf{Q}\mathbf{W} &= -b/(ad - bc), && (37)\\
-\mathbf{W}\mathbf{R}\mathbf{P}^{-1} &= -c/(ad - bc), && (38)
\end{aligned}$$

which gives the standard result for general 2×2 matrices.

3. Invertible Matrix Theory

Assume that $\mathbf{A} \in \mathbb{R}^{n \times n}$ and without using the invertible matrix theorem, prove the following:

3.1. **Spanning Sets.** If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{A}^{-1}$ exists, then the columns of $\mathbf{A}$ span $\mathbb{R}^n$.

If $\mathbf{A}$ is invertible then $\mathbf{A}$ has a pivot in each row, which implies that $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^n$. Thus, $\mathbf{A}$ maps onto $\mathbb{R}^n$ or the columns of $\mathbf{A}$ span $\mathbb{R}^n$.

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[8]Hint: Same as before but now it is easiest to check $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

3.2. **Pivot Structure.** If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{Ax} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^n$, then $\mathbf{A}$ is invertible.

Since $\mathbf{Ax} = \mathbf{b} \in \mathbb{R}^n$ has a solution for every $\mathbf{b}$ we know that $\mathbf{A}$ has a pivot for each row, which implies a pivot for each column. Thus, $\mathbf{A} \sim \mathbf{I}$ and $\mathbf{A}$ is invertible.

3.3. **Linear Independence.** If the matrix $\mathbf{A}$ is invertible, then the columns of $\mathbf{A}^{-1}$ are linearly independent.

If $\mathbf{A}$ is invertible then $\mathbf{A}^{-1}$ exists and is invertible. Since $\mathbf{A}^{-1}$ is invertible $\mathbf{A}^{-1} \sim \mathbf{I}$, which implies that every column of $\mathbf{A}$ has a pivot and the columns of $\mathbf{A}^{-1}$ are linearly independent.

3.4. **Free Variables I.** If the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, has more than one solution for some $\mathbf{b} \in \mathbb{R}^n$, then the columns of $\mathbf{A}$ do not span $\mathbb{R}^n$.

If $\mathbf{Ax} = \mathbf{b}$ has non-unique solutions for some $\mathbf{b}$ we know that $\mathbf{A}$ must not have a pivot for each column, which implies that there isn't a pivot for each row. Thus, $\mathbf{A}$ does not map onto $\mathbb{R}^n$ and its columns do not span $\mathbb{R}^n$.

3.5. **Free Variables II.** If the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, is inconsistent for some $\mathbf{b} \in \mathbb{R}^n$, then the equation $\mathbf{Ax} = \mathbf{0}$ has a non-trivial solution.

Inconsistency of $\mathbf{Ax} = \mathbf{b}$ implies that there isn't a pivot for each row but since $\mathbf{A}$ is square there isn't a pivot for each column. This implies the existence of free-variables and thus non-trivial solutions to the homogeneous problem.

3.6. **Linear Dependence.** If $\mathbf{A}$ is a square matrix with two identical columns then $\mathbf{A}^{-1}$ does not exist.

If two columns are the same then they form a linearly dependent set, which implies there isn't a pivot for each column. If these vectors are then placed into a square matrix then the matrix will have at least one column without a pivot. Thus, $\mathbf{A}$ cannot be row-reduced to the identity matrix and is therefore not invertible.

4. Matrix Decompositions

4.1. **LU Factorization.** Given,

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -1 & 5 \\ 3 & 7 & -2 & 9 \\ -2 & -3 & 1 & -4 \end{bmatrix}.$$

Determine the LU-Decomposition of the matrix $\mathbf{A}$ **and** check your result for $\mathbf{L}$ by multiplication of three elementary matrices.[9]
The $\mathbf{A}$ matrix row-reduces to the following echelon form,

$$\mathbf{A} \sim \begin{bmatrix} 1 & 4 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{U}, \tag{39}$$

by using row-steps given by the following elementary matrices,

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \tag{40}$$

whose inversions give our $\mathbf{L}$ matrix,

$$\mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1} = \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}. \tag{41}$$

[9]The matrix $\mathbf{U}$, found by three steps of row reduction on $\mathbf{A}$, will have two pivot columns. These two pivot columns are used to determine the first two columns of $\mathbf{L}_{3\times 3}$. The remaining column of $\mathbf{L}$ is equal the last column of $\mathbf{I}_3$.

4.2. **Spectral Factorization.** Suppose $\mathbf{A} \in \mathbb{R}^{3\times 3}$ admits a factorization $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where $\mathbf{P} \in \mathbb{R}^{3\times 3}$ is a invertible matrix and $\mathbf{D} \in \mathbb{R}^{3\times 3}$ is the diagonal matrix,[10]

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}. \tag{42}$$

Find a formula for $\lim\limits_{k\to\infty} \mathbf{A}^k$. [11]
Since $\mathbf{D}$ is a diagonal matrix we have that,

$$\lim_{k\to\infty} \mathbf{A}^k = \lim_{k\to\infty} \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1} \tag{43}$$

$$= \lim_{k\to\infty} \mathbf{P} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/2)^k & 0 \\ 0 & 0 & (1/3)^k \end{bmatrix} \mathbf{P}^{-1} \tag{44}$$

$$= \mathbf{P} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{P}^{-1} \tag{45}$$

4.3. **QR Factorization.** Suppose that $\mathbf{A} = \mathbf{Q}\mathbf{R}$ where $\mathbf{Q}, \mathbf{R} \in \mathbb{R}^{n\times n}$ are invertible matrices and $\mathbf{R}$ is upper-triangular while $\mathbf{Q}$ is such that $\mathbf{Q}^{\mathsf{T}}\mathbf{Q} = \mathbf{I}$. Show that for each $\mathbf{b} \in \mathbb{R}^n$ the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution and without using $\mathbf{R}^{-1}$ find formulas for calculating $\mathbf{x}$.
We have that $\mathbf{A}$ is the product of invertible matrices therefore $\mathbf{A}$ is invertible and there exists exactly one solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ for each $\mathbf{b} \in \mathbb{R}^n$. Moreover, using this decomposition we can reduce the problem to an equivalent problem that is already in echelon form,

$$\mathbf{A}\mathbf{x} = \mathbf{Q}\mathbf{R}\mathbf{x} = \mathbf{b} \iff \mathbf{Q}^{\mathsf{T}}\mathbf{Q}\mathbf{R}\mathbf{x} = \mathbf{R}\mathbf{x} = \mathbf{Q}^{\mathsf{T}}\mathbf{b}, \tag{46}$$

which is solved up to back-substitution steps.

4.4. **Singular Value Decomposition: Special Case.** Suppose that $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathsf{T}}$ where $\mathbf{U}, \mathbf{V} \in \mathbb{R}n \times n$ are invertible with the property that their transposes are their own inverses and $\mathbf{\Sigma}$ is a diagonal matrix with positive entries on the diagonal. Show that $\mathbf{A}$ is an invertible matrix and find a formula for $\mathbf{A}^{-1}$.

Since $\mathbf{\Sigma}$ is a diagonal matrix with strictly positive diagonal entries is has $n-$many pivots and is therefore invertible. Thus, $\mathbf{A}$ is written as the product of invertible matrices and is therefore invertible. Its inverse is given by the formula $\mathbf{A}^{-1} = (\boldsymbol{U}\mathbf{\Sigma}\boldsymbol{V}^{\mathsf{T}})^{-1} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^{\mathsf{T}}$, where $[\mathbf{\Sigma}]_{ij} = \sigma_i^{-1}\delta_{ij}$.

5. Determinants

5.1. **Determinants of Inversions.** Show that if $\mathbf{A}$ is invertible, then $\det(\mathbf{A}^{-1}) = \dfrac{1}{\det(\mathbf{A})}$.

By properties of determinants we have $\det(\mathbf{A}\mathbf{A}^{-1}) = \det(\mathbf{A})\det(\mathbf{A}^{-1}) = \det(\mathbf{I}) = 1$, which implies that $\det(\mathbf{A}^{-1}) = [\det(\mathbf{A})]^{-1}$.

5.2. **Determinants of Orthogonal Matrices.** Let $\mathbf{U}$ be a square matrix such that $\mathbf{U}^{\mathsf{T}}\mathbf{U} = \mathbf{I}$. Show that $\det(\mathbf{U}) = \pm 1$.

By properties of determinants we have $\det(\mathbf{U}\mathbf{U}^{\mathsf{T}}) = \det(\mathbf{U})\det(\mathbf{U}^{\mathsf{T}}) = \det(\mathbf{U})\det(\mathbf{U} = [\det(\mathbf{U}]^2 = \det(\mathbf{I}) = 1$, which implies that $\det(\mathbf{U}) = \pm 1$.

5.3. **Determinants of Similar Matrices.** Let $\mathbf{A}$ and $\mathbf{P}$ be square matrices such that $\mathbf{P}^{-1}$ exists. Show that $\det(\mathbf{P}\mathbf{A}\mathbf{P}^{-1}) = \det(\mathbf{A})$.
By properties of determinants we have $\det(\mathbf{P}\mathbf{A}\mathbf{P}^{-1}) = \det(\mathbf{P})\det(\mathbf{A})\det(\mathbf{P}^{-1}) = \det(\mathbf{A})$.

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[10] A diagonal matrix is a matrix that is both upper <u>and</u> lower triangular. That is $\mathbf{A} \in \mathbb{R}^{m\times n}$ is diagonal if and only if $[\mathbf{A}]_{ij} = 0$ for $i \neq j$.

[11] Hint: First find a formula for $\mathbf{A}^k$ using the spectral factorization. In this formula the exponent should only change the $\mathbf{D}$ matrix.

5.4. **Row-Operation Sanity Check.** Given the following for matrices:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}.$$

Calculate the determinants of the previous matrices by theorem 2.2.4. In each case, state the row-operation used on $\mathbf{A}$ to get to $\mathbf{B}, \mathbf{C}, \mathbf{D}$ and describe how it affects the determinant.

We note the following,

- $\mathbf{A} \sim \mathbf{B}$ by a row-swap
- $\mathbf{A} \sim \mathbf{C}$ by a row-scaling
- $\mathbf{A} \sim \mathbf{D}$ by a row-replacement

Calculating the determinants of each we get,

$$\det(\mathbf{A}) = ad - bc \tag{47}$$

$$\det(\mathbf{B}) = bc - ad = -\det(\mathbf{A}) \tag{48}$$

$$\det(\mathbf{C}) = kad - kbc = k(ad - bc) = k\det(\mathbf{A}) \tag{49}$$

$$\det(\mathbf{D}) = ad - bc = \det(\mathbf{A}), \tag{50}$$

which agrees with the rules discussed in class.

5.5. **Scaling Properties.** Find a formula for $\det(r\mathbf{A})$ where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $r \in \mathbb{R}$.

Since $r\mathbf{A}$ scales each row by a factor of r we can apply the scaling rule n-many times to get $\det(r\mathbf{A}) = r^n\det(\mathbf{A})$.

5.6. ***Vandermonde Determinant.*** Given,

$$\mathbf{A} = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}.$$

Show that the $\det(\mathbf{A}) = (c-a)(c-b)(b-a)$.[12]
This calculation is easiest done in conjunction with row-reduction. The following row-reduction,

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \underset{R3 \to R3 - R1}{\overset{R2 \to R2 - R1}{\sim}} \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} \overset{R3 \to R3 - \frac{(c-a)}{(b-a)}R2}{\sim} \begin{bmatrix} 1 & a & a^2 \\ 0 & (b-a) & b^2 \\ 0 & 0 & c^2 - a^2 - (c-a)(b^2-a^2)/(b-a) \end{bmatrix},$$

implies that,

$$\begin{aligned} \det(\mathbf{A}) &= 1 \cdot (b-a) \cdot \left(c^2 - a^2 - (c-a)\frac{b^2-a^2}{b-a}\right) \\ &= (b-a)\left((c-a)(c+a) - (c-a)\frac{(b-a)(b+a)}{b-a}\right) \\ &= (b-a)(c-a)\,(c+a-b+a) \\ &= (b-a)(c-a)\,(c-b) \end{aligned}$$

5.7. **Multi-linearity.** The determinant is not, in general, a linear mapping. That is, det: $\mathbb{R}^{n\times n} \to \mathbb{R}$ is **not**, in general, such that, $\det(\mathbf{A}+\mathbf{B}) = \det(\mathbf{A})+\det(\mathbf{B})$. The determinant is, in general, *multilinear.*[13] Show this for the domain $\mathbb{R}^{3\times 3}$ by verifying that $\det(\mathbf{A}) = \det(\mathbf{B}) + \det(\mathbf{C})$, where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are given as,[14]

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & u_1 + v_1 \\ a_{21} & a_{22} & u_2 + v_2 \\ a_{31} & a_{32} & u_3 + v_3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ a_{31} & a_{32} & u_3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{bmatrix}.$$

[12]Hint: It would be in your best interest to use row-reduction methods. This, of course, generalizes. `http://en.wikipedia.org/wiki/Vandermonde_matrix`

[13]A multilinear map is a mathematical function of several vector variables that is linear in each variable. That is, if all columns except one are fixed, then the determinant is a linear function of that one column. See `http://en.wikipedia.org/wiki/Multilinear_map` for more information.

[14]The easiest way to do this is by considering a cofactor expansion down the third column of $\mathbf{A}$. In this case the sums will appear as prefactors and distribution of multiplication over addition breaks the expansion into two expansions.

To verify this formula we compute the cofactor expansion down the third column of $\mathbf{A}$. Doing so gives,

$$\det(\mathbf{A}) = \sum_{i=1}^{3} a_{i3}(-1)^{i+3}\det\left(\mathbf{A}_{i3}\right) \tag{51}$$

$$= \sum_{i=1}^{3} \left(u_i + v_i\right)\left(-1\right)^{i+3}\det\left(\mathbf{A}_{i3}\right) \tag{52}$$

$$= \sum_{i=1}^{3} u_i(-1)^{i+3}\det\left(\mathbf{A}_{i3}\right) + \sum_{i=1}^{3} v_i(-1)^{i+3}\det\left(\mathbf{A}_{i3}\right) \tag{53}$$

$$= \sum_{i=1}^{3} u_i(-1)^{i+3}\det\left(\mathbf{B}_{i3}\right) + \sum_{i=1}^{3} v_i(-1)^{i+3}\det\left(\mathbf{C}_{i3}\right) \tag{54}$$

$$= \det(\mathbf{B}) + \det(\mathbf{C}) \tag{55}$$