

To get full credit, you must show all of your work.

1. Solve the initial value problems explicitly:

$$\frac{dy}{dt} - \tan(t)y = \cos(t), y(0) = 8, 0 \leq t < \frac{\pi}{2}.$$

$$\frac{dy}{dt} + (-\tan t)y = \cos t$$

$g(t)$ $b(t)$

u-sub.

$$u = \cos t, \quad du = -\sin t dt$$

$$u(t) = e^{\int -\tan t dt} = e^{\int -\frac{\sin t}{\cos t} dt} = e^{\ln |\cos t|} = |\cos t| = \cos t, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\cos t \frac{dy}{dt} + (\cos t)(-\frac{\sin t}{\cos t})y = \cos^2 t$$

$$\cos t \frac{dy}{dt} + (-\sin t)y = \cos^2 t$$

$$(y \cos t)' = \cos^2 t$$

$$y \cos t = \int \cos^2 t dt$$

Half-angle formula
 $\cos^2 t = \frac{1}{2}(1 + \cos(2t))$

$$y \cos t = \int \frac{1}{2}(1 + \cos(2t)) dt$$

$$y \cos t = \frac{t}{2} + \frac{1}{4} \sin(2t) + C$$

$$y = \frac{t}{2 \cos t} + \frac{\sin(2t)}{4 \cos t} + \frac{C}{\cos t} : \text{Explicit}$$

$$y(0) = 0 + 0 + C = 8, \quad C = 8$$

$$\boxed{\begin{aligned} y(t) &= \frac{t}{2 \cos t} + \frac{\sin(2t)}{4 \cos t} + \frac{8}{\cos t} \\ &= \frac{2t + \sin(2t) + 32}{4 \cos t} \end{aligned}} : \text{Explicit} \\ \text{+ particular}$$

2. Consider the differential equation, $ty' = t^2y - 2t^2$. Which two methods of solution can you use to solve this problem? Solve using both methods to check your answers.

$$ty' = t^2y - 2t^2$$

$$\frac{dy}{dt} = ty - 2t$$

Separation of variables

$$\frac{dy}{dt} = t(y-2)$$

$$\int \frac{1}{y-2} dy = \int t dt \quad (y \neq 2)$$

$$\ln|y-2| = \frac{t^2}{2} + C$$

$$|y-2| = e^C e^{\frac{t^2}{2}}$$

$$y-2 = \pm e^C e^{\frac{t^2}{2}}$$

$$y-2 = A e^{\frac{t^2}{2}}, \quad A = \pm e^C, \text{ or } (y=2 \text{ case})$$

$$\boxed{y = A e^{\frac{t^2}{2}} + 2}$$

Separation of variables
+ Integrating Factors

(You cannot use method of undetermined coefficients because $a(t) = t \neq 1 = \text{constant}$)

Integrating Factor:

$$\frac{dy}{dt} + (-t)y = -2t$$

$$u(t) = e^{\int -t dt} = e^{-\frac{t^2}{2}}$$

$$e^{-\frac{t^2}{2}} \frac{dy}{dt} + e^{-\frac{t^2}{2}} (-t)y = -2t e^{-\frac{t^2}{2}}$$

$$(e^{-\frac{t^2}{2}} y)' = -2t e^{-\frac{t^2}{2}}$$

$$\begin{aligned} u &= -\frac{t^2}{2} \\ du &= -t dt \end{aligned}$$

$$e^{-\frac{t^2}{2}} y = \int -2t e^{-\frac{t^2}{2}} dt$$

$$\begin{aligned} e^{-\frac{t^2}{2}} y &= 2 e^{-\frac{t^2}{2}} + C \\ y &= C (2 e^{-\frac{t^2}{2}} + C) \\ \boxed{y = 2 + C e^{\frac{t^2}{2}}} \end{aligned}$$

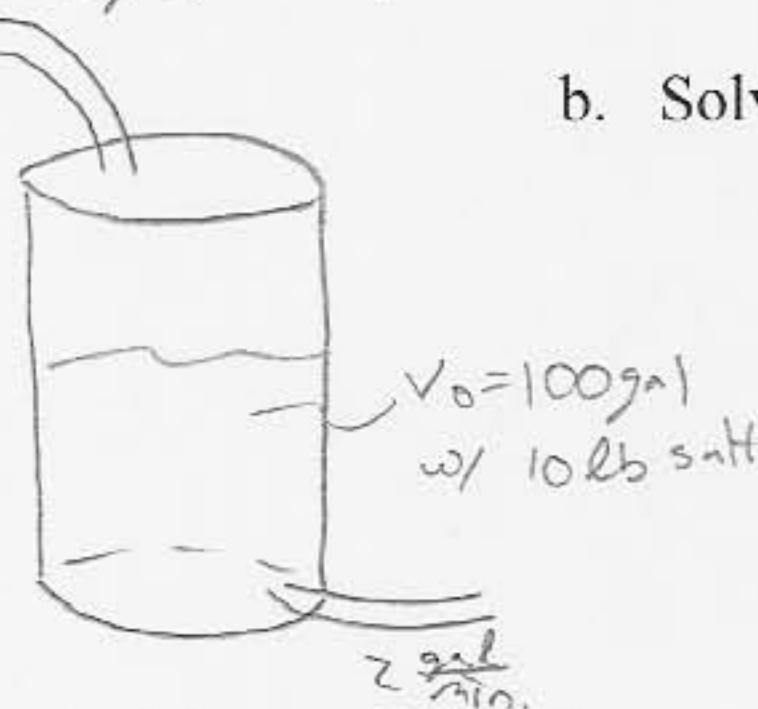
Same as above

3. A tank with a capacity of 400 gallons initially contains 100 gallons of water in which 10 pounds of salt is dissolved. Water containing 2 lb of salt per gallon is flowing into the tank at the rate of 4 gallons per minute. The well-stirred mixture leaves the tank at the rate of 2 gallons per minute.

- a. Write down the initial value problem that models the dynamics of the tank.

$$\frac{dy}{dt} = 2(4) - \frac{y}{100+2t}(2), \quad \boxed{\frac{dy}{dt} = 8 - \frac{y}{50+t}, \quad y(0)=10}$$

- b. Solve the initial value problem from (a).



t : time (min.)

y : amount of salt
in the tank (lb)

$\frac{dy}{dt}$: rate of change
of salt in
tank ($\frac{lb}{min}$)

V : Volume (gal)

$$V(t) = 100 + (4-2)t$$

$$V(t) = 100 + 2t$$

$$\frac{dy}{dt} + \left(\frac{1}{50+t}\right)y = 8$$

$$u(t) = e^{\int \frac{1}{50+t} dt} = e^{\ln|50+t|} = 50+t$$

$$(50+t) \frac{dy}{dt} + y = 8(50+t)$$

$$((50+t)y)' = 8(50+t)$$

$$(50+t)y = \int 8(50+t) dt$$

$$(50+t)y = 4(50+t)^2 + C$$

$$y = 4(50+t) + \frac{C}{(50+t)}$$

$$y(0) = 4(50) + \frac{C}{50} = 10$$

$$\frac{C}{50} = -190$$

$$C = -9500$$

$$y(t) = 4(50+t) - \frac{9500}{50+t}$$

- c. When will the tank start overflowing?

$$V(t) = 400$$

$$100 + 2t = 400, \quad t = 150 \text{ min.}$$

- d. How much salt is in the tank at the point of overflowing? Simplify your answer.

$$y(150) = 4(200) - \frac{9500}{200}$$

$$= \underline{\underline{800 - 47.5}}$$

$$= \underline{\underline{752.5 \text{ lb}}}$$

* Other equivalent forms of $y(t)$ are possible.

$$y(t) = \frac{8(100t+t^2+25)}{100+t}$$

4. Consider the predator-prey model defined by

$$\frac{dx}{dt} = -0.1x + 0.02xy$$

$$\frac{dy}{dt} = 0.2y - 0.025xy$$

- a. Which variable represents the prey and which represents the predator?

Why?

y : prey, interaction term $(-0.025xy)$ is negative

x : predator, interaction term $(0.02xy)$ is positive

- b. What are the equilibrium solutions of the system?

$$\frac{dx}{dt} = 0 \quad +$$

$$x(-0.1 + 0.02y) = 0$$

$$\underline{x=0} \quad -0.1 + 0.02y = 0$$

$$0.02y = 0.1$$

$$y = \frac{\frac{1}{10}}{\frac{2}{100}} = \frac{100}{20}$$

$$y = 5$$

$$\frac{dy}{dt} = 0$$

$$y(0.2 - 0.025x) = 0$$

$$\underline{y=0} : 0.2y = 0$$

$$\underline{y=0}$$

$$\underline{y=5} : 5(0.2 - 0.025x) = 0$$

$$0.2 - 0.025x = 0$$

$$x = \frac{0.2}{0.025} = \frac{2}{\frac{100}{1000}}$$

$$\underline{x=8}$$

5. Consider the differential equation $y'' + y' + 2y = 0$.

- a. Show that $y(t) = e^{-t/2} \cos\left(\frac{t\sqrt{7}}{2}\right)$ is a solution.

$$y'(t) = -\frac{1}{2}e^{-t/2} \cos\left(\frac{t\sqrt{7}}{2}\right) - \frac{\sqrt{7}}{2}e^{-t/2} \sin\left(\frac{t\sqrt{7}}{2}\right)$$

$$y''(t) = \frac{1}{4}e^{-t/2} \cos\left(\frac{t\sqrt{7}}{2}\right) + \frac{\sqrt{7}}{4}e^{-t/2} \sin\left(\frac{t\sqrt{7}}{2}\right) + \frac{\sqrt{7}}{4}e^{-t/2} \sin\left(\frac{t\sqrt{7}}{2}\right) - \frac{3}{4}e^{-t/2} \cos\left(\frac{t\sqrt{7}}{2}\right)$$

$$y'' + y' + 2y = -\frac{3}{2}e^{-t/2} \cos\left(\frac{t\sqrt{7}}{2}\right) + \frac{\sqrt{7}}{2}e^{-t/2} \sin\left(\frac{t\sqrt{7}}{2}\right) + \frac{1}{2}e^{-t/2} \cos\left(\frac{t\sqrt{7}}{2}\right) - \frac{\sqrt{7}}{2}e^{-t/2} \sin\left(\frac{t\sqrt{7}}{2}\right) + 2e^{-t/2} \cos\left(\frac{t\sqrt{7}}{2}\right) = 0 + 0 = 0$$

- b. Let $v = \frac{dy}{dt}$ and convert the second-order differential equation into a first-order system.

$$y'' + y' + 2y = 0$$

$$y'' = -y' - 2y$$

$$v = \frac{dy}{dt}$$

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = y'' = -y' - 2y$$

$$= -v - 2y$$

\Rightarrow

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -2y - v$$