PHGN 462 Homework 5

1) (From Pollack and Stump 11.13)

Consider the electromagnetic field

$$\vec{E}(x,y,t) = E_0 \cos\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi y}{L}\right)\sin(\omega t)\,\hat{k}$$

$$\vec{B}(x,y,t) = B_0\left[-\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)\hat{\imath} + \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi y}{L}\right)\hat{\jmath}\right]\cos(\omega t)$$

This is an example of a standing wave. Standing waves in cavities show up in a lot of practical places; not the least of which is a laser cavity. This one in particular might be a wave in a long rectangular box stretching out in the *z* direction, with walls of length *L* in the *x* and *y* directions.

a) Show that this field satisfies the Maxwell equations in vacuum if $\omega = \frac{\sqrt{2}\pi c}{L}$ and $B_0 = \frac{E_0}{\sqrt{2}c}$. Notice in particular that our old familiar $E = cB$ doesn't apply here – that result was derived for plane waves, and this isn't one.

b) The problem statement says we have a box of side lengths *L* in *x* and *y*, but doesn't say where the origin is. Maybe the origin sits at the center of the *x-y* cross section of the box. Or maybe the origin sits at one corner of said cross section. Put another way, maybe the *x* and *y* domains are $\{-\frac{L}{2}, \frac{L}{2}\}$, or maybe they're $\{0, L\}$.

If we assume the sides of the box are conducting, one of these options works physically and one doesn't. Looking at the basic boundary conditions for *E* and *B*, figure out what the *x* and *y* domains must be for these fields.

2) (From Pollack and Stump 13.23)

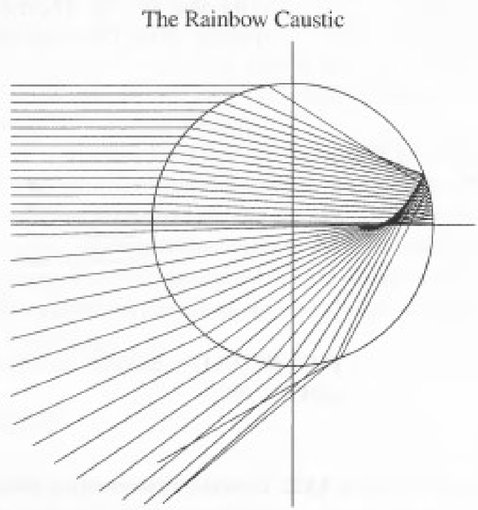


FIGURE 13.13 Illustration of how light refraction and reflection in a water droplet causes a rainbow. Light incident from the left is refracted on entering the spherical drop, reflects partially from the back surface, and is refracted on leaving the front surface. (Only rays incident above the midplane are shown.) The caustic—the region where the exiting rays are most concentrated—is the rainbow. The colors of the rainbow are the result of dispersion; the angle of the caustic varies with wavelength.

Figure 13.13 shows the light scattering process that creates the primary rainbow. Light rays at varying impact parameter refract into a spherical water drop, reflect from the back surface, and refract out of the drop. (In the figure, only the rays entering the upper half of the drop are shown. The rays shown are the rays that would reach the ground.) At a scattering angle of 42 degrees there is a concentration of scattered rays, called the caustic, and that somewhat more intense scattered light is the rainbow.

a) Explain why the ordering of colors (ROYGBIV) is red at the outer edge of the arc, and violet at the inner edge. You'll need to look up and take into account the manner in which the index of refraction of water varies with wavelength.

b) A secondary rainbow, in which the order of colors is reversed, is sometimes visible at a higher angle than the primary. Explain this second arc.

c) Explain why the area inside the primary rainbow is brighter than the area outside. See, for example, the region inside the arc in this picture:

https://upload.wikimedia.org/wikipedia/commons/c/c0/Rainbow_02.jpg

Note that "explain" doesn't mean copy text from Wikipedia with a couple words changed. I want to see diagrams and geometry and stuff. Roughed out, at least. Also note that to figure this out, you'll need to look up how the index of refraction of water varies with wavelength.

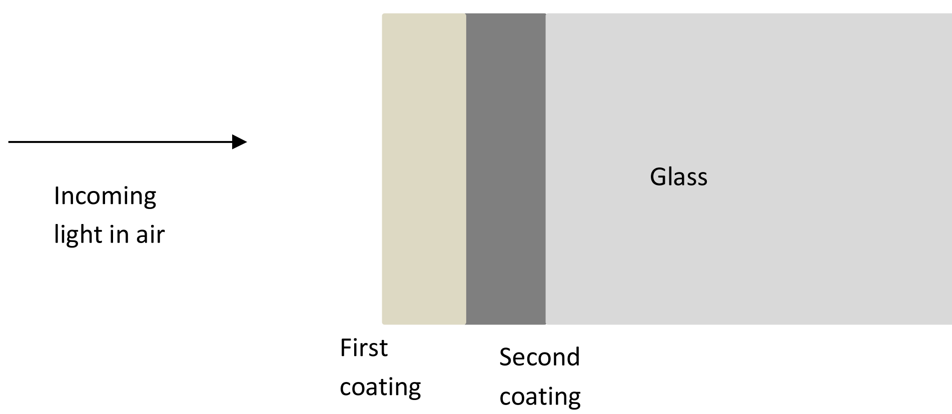
3) (Based on Pollack and Stump 13.5 – also, read problem 4 before you start this one)

Consider light incident normally on a plate of glass with thickness a, with air on either side of the plate. You can treat the air as having index of refraction 1, and the glass as having index 1.5.

a) Write **E**(x,t) and **B**(x,t) in the three regions, with x being the propagation axis and $\hat{\jmath}$ being the direction in which the E-field is polarized. Write the four non-trivial boundary conditions on the wave amplitudes (two at each interface).

b) Solve for the transmission coefficient T, i.e., the ratio of transmitted intensity to incident intensity. Plot T as a function of $k*a$, where k is the magnitude of the incident wave vector. While it is technically possible to do this analytically for this problem, I strongly recommend you do it numerically. Interpret the resulting graph physically, and tell me why it has the period it has.

4) If you set up problem 3 in a really general, symbolic way using good Mathematica code, you have the foundation you need to solve more complicated multilayer problems fairly quickly. Let's try something that's pushing in the direction of a realistic industry problem.



The above diagram is a crude picture of glass with two dielectric coatings on it. It could be a lens, a mirror, or a bunch of other things. Let's use materials that are common in actual practice. The substrate is BK7 glass, very common in optics on account of being relatively low-dispersion, which means that different colors of visible light see more-or-less the same index of refraction, 1.52. One of the coatings is made of MgF_2 (magnesium fluoride), with approximate index 1.37 for visible wavelengths. The other is made of Al_2O_3 (aluminum oxide), with index 1.77. Both of these materials are very durable and scratch-resistant, and make great coatings. As a side note, aluminum oxide is the basis for gems such as sapphire and ruby.

This is a physical setup with four regions and three interfaces. Note that we're considering the output end to be in the glass, *not* the air on the far end of the glass. We're going to solve for the transmission coefficient of this system as a function of incoming wavelength and coating thickness, considering the possibility that the coatings might be of unequal thickness.

a) Write down **E**(x,t) and **B**(x,t) in the four regions, letting $\hat{\jmath}$ be the polarization direction for **E**. Then write down the six relevant boundary conditions on those fields. This should be at most a minor extension of the work you did in (1a).

b) Use Mathematica (or whatever) to get the transmission coefficient for the system as a function of thicknesses a and b and incident wavelength λ. Repackage the variables in whatever way you find convenient (for example, if you'd rather work with wavevector k, or some defined combo like $x = k*a$, go nuts). Pick some interesting, specific values for a, b, and λ and graph T as a function of λ (or k, or $k*a$, or whatever you find appropriate). This, also, should be pretty much re-use of what you did in problem 1.

c) Now let's get interesting. People tailor multilayer dielectric coatings such as this one towards specific design constraints. For example, a client might need an optic that simultaneously minimizes reflection of 532 and 650nm light, presumably cutting out other wavelengths in the process. A company that makes custom coated optics wouldn't limit themselves to two layers using two materials, but for the sake of keeping it "easy" let's say we have the previously specified system and we want to choose values for a and b that minimize reflections at 532nm and 650nm. Don't sweat what happens at other wavelengths.

Optimization is kind of a tricky business, and I'm not going require that you do it a certain way. This is where you get to exercise a little creativity. Find the best possible a and b for the given parameters any way you please.

Make sure that when you're writing up this problem, you tell me what you did and show me a graph of your optimal solution. You get full credit for coming up with any remotely decent solution and a clear explanation of what you've done.