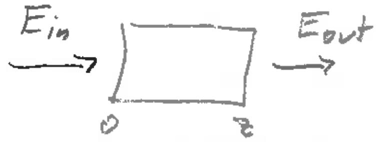
Dispersive pulse propagation

For linear propagation, the effect of a dispersive system is easiest to calculate in the frequency domain



$$E_{out}(\omega) = e^{i\frac{\omega}{c}n(\omega)z}\, E_{in}(\omega)$$

for example where dispersion comes from refractive index $n(\omega)$

real optical materials (glasses, crystals) have complicated $n(\omega)$. Typically use empirical functions $n(\lambda)$: Sellmeier eqns.

Away from absorption $n(\omega)$ is real. To make computation easier, Taylor expand spectral phase around ω_0 carrier

$$\phi(\omega) \approx \phi(\omega_0) + (\omega-\omega_0)\left.\frac{\partial\phi}{\partial\omega}\right|_{\omega_0} + \frac{1}{2!}(\omega-\omega_0)^2\left.\frac{\partial^2\phi}{\partial\omega^2}\right|_{\omega_0} + \cdots$$

notation: $= \phi_0 + \Delta\omega\,\phi_1 + \frac{1}{2}\Delta\omega^2\phi_2 + \ldots$

for a given thickness L $\qquad \phi(\omega) = \frac{\omega}{c}n(\omega)L$

$$\phi_1 \equiv \left.\frac{\partial\phi}{\partial\omega}\right|_{\omega_0} = \frac{L}{c}\left.\left(n(\omega) + \omega\frac{\partial n}{\partial\omega}\right)\right|_{\omega_0}$$

Sellmeier eqns are represented as $n(\lambda)$

$$\frac{\partial n}{\partial\omega} = \frac{\partial\lambda}{\partial\omega}\frac{\partial n}{\partial\lambda} = -\frac{2\pi c}{\omega^2}\frac{dn}{d\lambda}$$

Interpretation:

$E_{out} = e^{i\phi_0}E_{in}$ $\qquad \phi_0$ = constant phase shift

same in ω, t spaces

$$E_{out}(\omega) = E_{in}(\omega) e^{i(\omega-\omega_0)\phi_1}$$

spectral shape is centered on ω_0 $\mathcal{F}^{-1}\{E_{in}(\omega-\omega_0)\} = A(t) e^{-i\omega_0 t}$

$$\mathcal{F}^{-1}\left\{E_{in}(\omega-\omega_0) e^{i(\omega-\omega_0)\phi_1}\right\} = A(t-t_0) e^{-i\omega_0(t-t_0)}$$

by shift theorem

$t_0 = \phi_1$ = <u>group delay</u>

general form of group delay

$$\phi_1(\omega) = \frac{\partial\phi}{\partial\omega} = \frac{\partial}{\partial\omega}(kL) = L \Big/ \frac{\partial\omega}{\partial k} = L/v_g$$

group velocity

Note $\phi_1(\omega)$ varies with ω

$\rightarrow$ dispersion, changes in pulse shape.

$$\phi_2 = \frac{\partial}{\partial\omega}\phi_1(\omega) \quad \text{group delay dispersion}$$

Gaussian pulse example : ϕ_0, ϕ_1, ϕ_2

$$E_{out}(\omega) = E_{in}(\omega) e^{i(\phi_0 + (\omega-\omega_0)\phi_1 + \frac{1}{2}(\omega-\omega_0)^2\phi_2)}$$

$$E_{in}(t) = E_0 e^{-t^2/\tau_0^2} e^{-i\omega_0 t} \rightarrow E_{in}(\omega) = E_0\sqrt{\pi}\,\tau_0 e^{-(\omega-\omega_0)^2\frac{\tau_0^2}{4}}$$

let $\omega' = \omega - \omega_0$ $\qquad \phi_0 = \frac{\omega_0}{c} n(\omega_0) L = k_0 L$

$$E_{out}(\omega') = \underbrace{E_0\sqrt{\pi}\,\tau\, e^{ik_0 L}}_{\text{constants}} \underbrace{e^{-\omega'^2(\frac{\tau_0^2}{4} - i\phi_2/2)}}_{F(\omega')} e^{i\omega'\phi_1}$$

if $f(t) = \mathcal{F}^{-1}_{\omega'}\{F(\omega')\}$ then $\mathcal{F}^{-1}\{F(\omega') e^{i\omega'\phi_1}\} = f(t-\phi_1)$

$F(\omega')$ is a <u>gaussian</u> with a complex width

let $\frac{\tau_0^2}{4} - i\frac{\phi_2}{2} = \frac{\tau_0^2}{4}(1 - i\Gamma) \qquad \Gamma = \frac{2\phi_2}{\tau_0^2}$

$$= \frac{\tau'^2}{4}$$

$$f(t) = \mathcal{F}^{-1}\left\{ e^{-\omega'^2 \tau'^2/4} \right\} = \frac{1}{\sqrt{\pi}\,\tau'} e^{-t^2/\tau'^2}$$

$$E_{out}(t) = E_0 e^{i(k_0 L - \omega_0 t)} \frac{\tau}{\tau'} e^{-(t-\phi_1)^2/\tau'^2}$$

interpretation: look at intensity

$$I_{out}(t) \propto |E_{out}(t)|^2 \propto \frac{\tau^2}{|\tau'|^2} e^{-(t-\phi_1)^2/\tau'^2} e^{-(t-\phi_1)^2/\tau_1'^{*2}}$$

$$|\tau'|^2 = \tau_0^2(1+\Gamma^2)$$ output pulse broadens

$$\tau_{out}(z) = \tau_0\sqrt{1 + \frac{4\phi_2^2(z)}{\tau_0^4}}$$

recall that for a transform-limited pulse

$$\left.\begin{aligned} &\tau_0 \Delta\omega = 2 \\ \text{and} \quad &\phi_2 = \frac{\partial^2 k}{\partial \omega^2} z \equiv k_2 z \end{aligned}\right\} \quad \frac{4\phi_2^2}{\tau_0^4} = \frac{\Delta\omega^4}{4} k_2^2 z^2$$

let $z_d = \frac{\Delta\omega^2}{2} k_2$ = dispersion length

$$\tau_{out}(z) = \tau_0\sqrt{1 + z^2/z_d^2}$$ just like a Gaussian beam!

phase: quadratic phase in spectral domain

let $T = t - \phi_1$ (grid moves at group velocity)

$$\exp\left[-\frac{T^2}{\tau_0^2}\frac{1}{1-i\Gamma}\right] = \exp\left[-\frac{T^2(1+i\Gamma)}{\tau_0^2(1+\Gamma^2)}\right]$$

amplitude $e^{-T^2/\tau_{out}^2(z)}$

phase: $-\frac{T^2\,\Gamma(z)}{\tau_{out}^2(z)}$ quadratic <u>also</u>.

Pulse structure

the field can be represented in t- or ω-space.

ω-space $\quad E(\omega) = A(\omega-\omega_0)\, e^{i\phi(\omega-\omega_0)}$

$\phi(\omega)$ = spectral phase

$\phi'(\omega)$ = group delay = arrival time of freq. ω

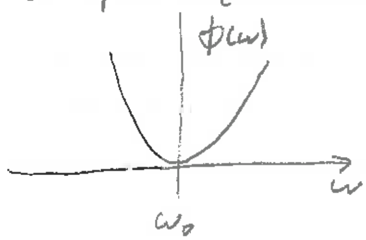
$= \tau_g$

t-space $\quad E(t) = A(t)\, e^{i\phi(t)}\, e^{-i\omega_0 t}$ $\quad$ t is referenced to pulse peak.

$\phi(t)$ = temporal phase

$-\frac{\partial\phi}{\partial t}$ = instantaneous frequency = ω_{inst}

chirp: quadratic spectral phase → linear g.d. with ω



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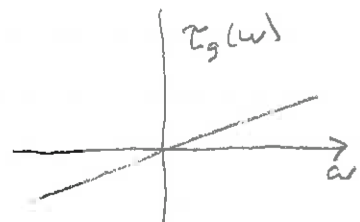
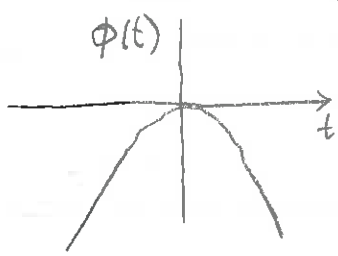


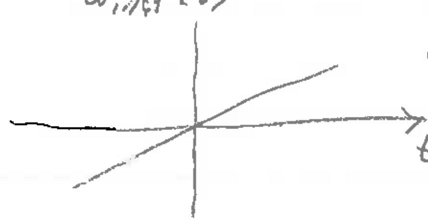
→ +'ve chirp

negative τ_g arrives <u>earlier</u>

Gaussian pulse with ϕ_2 but no higher order.

→ Gaussian w/ quadratic temporal phase.





also positive chirp.

But for higher-order phase, $\phi(t)$ and $\phi(\omega)$ aren't corresponding.

Higher-order effects

3rd-order dispersion

- suppose we use a zero-dispersion fiber

next higher-order term is 3rd-order:

$$\partial_z A = \frac{1}{6} k_3 \partial_t^3 A(t) = i \frac{1}{6} k_3 (\omega-\omega_0)^3 A(\omega)$$

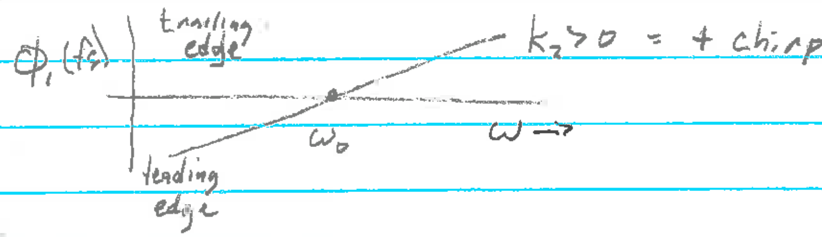
$\rightarrow$ spectral phase is $\phi(\omega) = \frac{1}{6} k_3 (\omega-\omega_0)^3 L$

group delay is $\frac{\partial \phi}{\partial \omega} = \frac{1}{2} k_3 (\omega-\omega_0)^2 L \equiv \phi_1(\omega)$

this represents the transit time as fcn of ω

e.g. linear chirp = quadratic phase

$\phi(\omega) = \frac{1}{2} k_2 (\omega-\omega_0)^2 L \rightarrow \phi_1 = k_2(\omega-\omega_0) L$

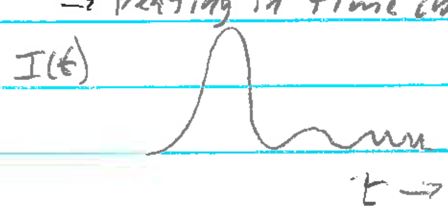


3rd order

here, pulse leads w/ central freq. ω_0

trails w/ mix of $\omega > \omega_0$ and $\omega < \omega_0$

$\rightarrow$ beating in time domain:



$\rightarrow$ asymmetry in SPM spectrum