

Linear Independence - Matrix Spaces - Vector Spaces

1. Determine the values of h for which the vectors are linearly dependent.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$$

2. Given,

$$\mathbf{A} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

- (a) Is \mathbf{w} in the column space of \mathbf{A} ? That is, does $\mathbf{w} \in \text{Col } \mathbf{A}$?
- (b) Is \mathbf{w} in the null space of \mathbf{A} ? That is, does $\mathbf{w} \in \text{Nul } \mathbf{A}$?

3. Given,

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}.$$

Determine:

- (a) The basis and dimension of $\text{Nul } \mathbf{A}$.
- (b) The basis and dimension of $\text{Col } \mathbf{A}$.
- (c) The basis and dimension of $\text{Row } \mathbf{A}$.
- (d) What is the Rank of \mathbf{A} ?

4. Let,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

- (a) How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?
- (b) How many vectors are in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?
- (c) Is \mathbf{w} in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

5. Given,

$$my'' + ky = 0, \quad m, k \in \mathbb{R}. \quad (1)$$

- (a) Show that $y_1(t) = \cos(\omega t)$ and $y_2(t) = \sin(\omega t)$, where $\omega = \sqrt{\frac{k}{m}}$, are solutions to the ODE.
- (b) Show that any function in $\text{Span}\{y_1, y_2\}$ is a solution to the ODE.¹

¹Since we know from differential equations that the only solutions to (1) are 0, y_1, y_2 we can conclude that the space of solutions to (1) forms a vector space whose basis is y_1 and y_2 .

Homework 3-Solutions

1. Using $\vec{v}_1, \vec{v}_2, \vec{v}_3$ construct the matrix,

$$V = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = \begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ -3 & 8 & h \end{bmatrix}$$

If the columns of V are linearly independent if and only if $V\vec{x} = \vec{0}$ has only the trivial solution. Thus

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ -3 & 8 & h & 0 \end{array} \right] \xrightarrow{\substack{R3=3R3+R1 \\ R2=R1+R2}} \sim \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -7 & 3+h & 0 \end{array} \right] \xrightarrow{R3=2R3+7R2} \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2h+20 & 0 \end{array} \right]$$

Shows that $V\vec{x} = \vec{0}$ has nontrivial solutions for
 $2h+20=0 \Leftrightarrow h = -10$

Hence, if $h = -10$ then $V\vec{x} = \vec{0}$ has nontrivial solutions and
if $V\vec{x} = \vec{0}$ has nontrivial solutions then the columns of
 V are linearly dependent.

5. Let,

$$A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

a. If $\vec{w} \in \text{Col } A$ then $\vec{w} \in \text{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$. Check by Row Reduction.

$$[A \mid \vec{w}] = \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{array} \right] \sim \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 0 & 20 & 10 & 20 \\ 0 & -2 & -1 & -2 \end{array} \right] \sim$$

$$\sim \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 0 & 20 & 10 & 20 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow x_3 \text{ is free}$$

and x_1, x_2 are uniquely determined in terms of x_3 .

Thus

There exists x_1, x_2, x_3 s.t. $x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 = \vec{w}$ and $\vec{w} \in \text{Col } A$.

b. $A \vec{w} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

thus $\vec{w} \in \text{Nul } A$.

$$3. A = \left[\begin{array}{ccccc} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{array} \right] \sim \left[\begin{array}{ccccc} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \end{array} \right] \sim$$

$R_2 = R_2 + R_1$
 $R_3 = R_3 - 2R_1$
 $R_4 = R_4 + R_1$

$$\sim \left[\begin{array}{ccccc} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \end{array} \right] \sim \left[\begin{array}{ccccc} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right] \sim$$

$R_3 = R_3 + R_2$
 $R_4 = R_4 + R_3$

$$\sim \left[\begin{array}{ccccc} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] = B$$

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a. $B \Rightarrow A\bar{x} = c$ has the general solution set,

$$x_4 = -3x_5$$

$$x_3 = (x_4 - x) = (\frac{-3x_5 - x}{3}) = \frac{-1}{3}x_5$$

$$x_1 = \frac{1}{2}(3x_2 - 6x_3 - 2x_4 - 5x_5) = \frac{1}{2}(3x_2 - 6(\frac{-1}{3}x_5) - 2(-3x_5) - 5x_5) = \frac{1}{2}(3x_2 + 8x_5 + 6x_5 - 5x_5) = \frac{3}{2}x_2 + \frac{9}{2}x_5$$

$x_2 = \text{free}$

$$x_5 = \text{free.} \Rightarrow \bar{x} = x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix}, x_2, x_5 \in \mathbb{R}$$

Thus the Basis for $\text{Nul } A$ is

$$B_{\text{Nul}} = \left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \right\}$$

and $\dim(\text{Nul } A) = \dim B_{\text{Nul}} = 2$

- b. B \Rightarrow That the basis for the column space of A is the pivot columns $\vec{a}_1, \vec{a}_3, \vec{a}_4$ of A.

$$B_{\text{Col } A} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \right\}$$

and

$$\dim(\text{Col } A) = \dim B_{\text{Col } A} = 3$$

- c. B \Rightarrow The Basis for $\text{Row } A$ is given as

$$B_{\text{Row } A} = \left\{ [2, -3, 6, 2, 5], [0, 0, 3, -1, 1], [0, 0, 0, 1, 3] \right\}$$

$$\dim(\text{Row } A) = \dim B_{\text{Row } A} = 3.$$

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5) a) $y_1(t) = \cos(\omega t) \Rightarrow y_1''(t) = -\omega^2 \cos(\omega t)$

$$y_2(t) = \sin(\omega t) \Rightarrow y_2''(t) = -\omega^2 \sin(\omega t)$$

$$\Rightarrow my_1'' + ky_1 = -m\omega^2 \cos(\omega t) + k \cos(\omega t) =$$

$$= \cos(\omega t)(-m\omega^2 + k) = \cos(\omega t)(-k + k) = 0$$

and

$$my_2'' + ky_2 = \sin(\omega t)(-m\omega^2 + k) = 0$$

b) Let $y \in \text{span}\{y_1, y_2\}$ then

$$y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

and

$$my'' + ky = C_1 \cos(\omega t)(-\omega^2 m + k) + C_2 \sin(\omega t)(-\omega^2 m + k) = 0$$

thus a fxn in $\text{span}\{y_1, y_2\}$ are soln.

4)

a) There are 3 vectors in

$$V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$$

b) There are ∞ -many vectors in
Span V

Each is of the form,

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 \quad \text{where } c_1, c_2, c_3 \in \mathbb{R}$$

c) If $\vec{w} \in \text{Span } V$ then

$$\vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

which is the same as ask
if

$$V \vec{c} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 3 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \vec{w}$$

has a soln.

$$[V | \vec{w}] = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array} \right] \xrightarrow{R3=R3+R1} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 5 & 10 & 5 \end{array} \right] \xrightarrow{R3=R3-5R2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Since there are 3 pivots there is a soln.

Since there is a soln we can conclude that

$$\vec{w} \in \text{Span } V$$