Kinematics and Dynamics

Motions in Space as it's Described by Linear Algebra

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Scott Strong

sstrong@mines.edu

Colorado School of Mines

Overview/Keywords/References

Advanced Engineering Mathematics **Slide Set Three**

Kinematics and Dynamics: Rigid Bodies, Fluid Fields

Examples: Rotations, Infinitesimal Generators, Vorticity

- See Also:
  - Linear Algebra, Peter D. Lax, Wiley, 1997
  - Wikipedia : Peter David Lax (born 1 May 1926 in Budapest, Hungary) is a mathematician working in the areas of pure and applied mathematics. He has made important contributions to integrable systems, fluid dynamics and shock waves, solitonic physics, hyperbolic conservation laws, and mathematical and scientific computing, among other fields. Lax is listed as an ISI highly cited researcher.

Abstract

Linear algebra is an old mathematical topic dating back to the 1600's and in its most abstract setting, vector spaces, characterizes the study of linear spaces, which is as complete of a theory as one could hope. Linear theory is applicable to linear and nonlinear phenomenon and it is almost impossible to avoid when working in either category.

- Linearization : $\mathbf{Y}' = \mathbf{f}(\mathbf{Y}) \Rightarrow \tilde{\mathbf{Y}}' = \mathbf{J}(\mathbf{f}(\mathbf{Y}_0))\tilde{\mathbf{Y}}$ where $\mathbf{Y}_0$ is a fixed point of the nonlinear problem and **J** is the Jacobian of **f**.
- Finite Difference Methods : $u_x \approx \frac{u(x_{i+1},t)-u(x_i,t)}{\Delta x}$, for $i = 0, 2, 3, \ldots, n-1$.
- Galerkin's Method : Assume that a solution to a problem exists in an infinite-dimensional linear space and search for its expansion in a finite-dimensional subspace.

Lecture Outline

- Motion of Rigid Bodies : When studying rigid body dynamics in a local coordinate system then special unitary matrices define rotations of the body. It is possible, using the formalism of linear algebra, to define an instantaneous axis/angle of rotation for this body.
- Kinematics of Fluid Flow : A fluidic particle is critically defined as being not rigid. However, the previous formalism can be applied to fluids andf one finds that the instantaneous angular velocity is proportional to the fields curl. Thus, a irrotational fluid is one that has no curl. Extending these concepts one can show that the instantaneous compression of the field is given by its divergence. Consequently, an incompressible fluid is one whose divergence is zero.

Before We Begin

Quote of Slide Set Three
Those evil-natured robots they're programmed to destroy us. She's gotta be strong to fight them so she's taking lots of vitamins.
The Flaming Lips : Yoshimi Battles the Pink Robots, Pt. 1 (2002)

Unitary Matrices

- Unitary Matrix : A matrix $\mathbf{U} \in \mathbb{C}^{m \times n}$ is called unitary if $\mathbf{U}^{\mathsf{H}}\mathbf{U} = \mathbf{I}$.
  - The definition implies,

$$\left[\mathbf{U}^{\mathsf{H}}\mathbf{U}\right]_{ij} = \mathbf{u}_i^{\mathsf{H}}\mathbf{u}_j = \langle \mathbf{u}_i, \mathbf{u}_j \rangle = \delta_{ij} = [\mathbf{I}]_{ij} \,, \tag{1}$$

    meaning that the columns of $\mathbf{U}$ form an orthonormal set.
  - The set of all matrices of this type forms a group, under matrix multiplication, called the unitary group, $U(n)$.
  - If the matrix is real then it is called orthogonal.
  - If the matrix is also square then the columns form an orthonormal basis for $\mathbb{R}^n$ and $\mathbf{U}\mathbf{U}^{\mathsf{T}} = \mathbf{U}^{\mathsf{T}}\mathbf{U} = \mathbf{I}$.

Unitary Transformations of $\mathbb{R}^n$

If $\mathbf{U} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a unitary matrix then:

- Angle Invariance : $\langle \mathbf{Ux}, \mathbf{Uy} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$
- Length Invariance : $||\mathbf{Ux}|| = ||\mathbf{x}||$
- Isometry : The mapping $\mathbf{U}$, into $\mathbb{R}^n$, preserves distance.

$$\mathbf{R}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad \mathbf{R}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

$$\mathbf{R}_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{U} = \mathbf{R}_2(\pi/4) = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}.$$

Unitary Transformations of $\mathbb{R}^3$

Theorem (Euler): A nontrivial isometry **U** of three-dimensional real Euclidean space with $\det(\mathbf{U}) = 1$ is a rotation and has a uniquely defined axis and angle of rotation. Moreover, **U** has exactly one eigenvalue equal to one.

- Proof : See Lax page 141.
- Key Point : If the eigenvectors of a matrix form a basis for $\mathbb{R}^3$ then $\mathbf{Ay} = \mathbf{A}(c_i\mathbf{x}_i) = c_i\lambda_i\mathbf{x}_i$ and we conclude that the transformation of **y**, relative to the eigenvector basis, is the linear combination of scalings in each eigendirection. Thus, each rotation matrix has a unique eigenvector for which the transformation does not alter the input vector.

Question : What can be said about the angle/axis of rotation without knowing the form of the rotation matrix?

Trace Formula

Recall the definition of the trace of a square matrix.

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}, \;\; \mathbf{A} \in \mathbb{R}^{n\times n} \tag{2}$$

Some properties of the trace for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$:

- Linearity : $\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})$
- Commutivity : $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$
- Invariance : If $\mathbf{A} = \mathbf{PBP}^{-1}$ then $\text{tr}(\mathbf{A}) = \text{tr}(\mathbf{B})$.

$$\text{tr}(\mathbf{A}) = \text{tr}(\mathbf{PBP}^{-1}) = \text{tr}((\mathbf{PB})\mathbf{P}^{-1}) = \text{tr}((\mathbf{P}^{-1}\mathbf{P})\mathbf{B}) = \text{tr}(\mathbf{IB}) = \text{tr}(\mathbf{B})$$

Key Point: A matrix has the same trace regardless of the basis it is represented in. From this we have, $\text{tr}(\mathbf{R}_i) = 1 + 2\cos(\theta)$.

Infinitesimal Generator of the Motion

Let $\theta = \theta(t)$ and consider the dynamic rotations of a curvilinear coordinate system whose origin corresponds to the first-moment of a rigid body such that $\mathbf{U}(0) = \mathbf{I}$. We note,

$$\frac{d}{dt}\left[\mathbf{U}\mathbf{U}^{\mathsf{T}}\right] = \mathbf{U}_t\mathbf{U}^{\mathsf{T}} + \mathbf{U}\mathbf{U}_t^{\mathsf{T}} \quad (3)$$

$$= \mathbf{A} + \mathbf{A}^{\mathsf{T}} = \frac{d}{dt}\mathbf{I} = 0, \quad (4)$$

where $\mathbf{A} = \mathbf{U}_t\mathbf{U}^{\mathsf{T}}$ is a skew-symmetric matrix, $\mathbf{A}^{\mathsf{T}} = -\mathbf{A}$. By this definition we have the system of differential equations,

$$\mathbf{U}_t = \frac{d\mathbf{U}}{dt} = \mathbf{A}(t)\mathbf{U}(t), \quad (5)$$

where $\mathbf{A}(t)$ is called an *infinitesimal generator* of the motion, which is independent of the reference time.

Evolution Equation

There is no general technique that solves the previous dynamical system. The general study of dynamical systems leads to fundamental problems involving chaotic dynamics and fractal geometry. Though there is a fair bit known about these systems here are some quick and important results.

- Constant Linear Case : If $\mathbf{A} \in \mathbb{R}^{3\times 3}$ is not a function of time then one has access to the matrix exponential $\mathbf{U}(t) = e^{\mathbf{A}t}$. This can the be studied through a diagonalization of $\mathbf{A}$, which leads to an eigenfunction decomposition.
- Tidy Time Dependence : If the time-dependent $\mathbf{A}$ matrix commutes with itself at every point in its domain, $[\mathbf{A}(t), \mathbf{A}(s)] = 0$ for all t and s, then $\mathbf{U}(t) = e^{\int_0^t \mathbf{A}(s)ds}$.

Axis of Rotation - Part I

Recall that Euler says that for every rotation matrix there exists a unique instantaneous axis of rotation, $\mathbf{f}(t)$, such that $\mathbf{U}(t)\mathbf{f}(t) = \mathbf{f}(t)$. Furthermore,

$$\begin{aligned}
\frac{d}{dt}\left[\mathbf{U}(t)\mathbf{f}(t)\right]_{t=0} &= \mathbf{U}_t(0)\mathbf{f}(0) + \mathbf{U}(0)\mathbf{f}_t(0) && (6)\\
&= \mathbf{U}_t(0)\mathbf{f}(0) + \mathbf{I}\mathbf{f}_t(0) && (7)\\
&= \mathbf{A}(0)\mathbf{U}(0)\mathbf{f}(0) + \mathbf{f}_t(0) && (8)\\
&= \mathbf{A}(0)\mathbf{I}\mathbf{f}(0) + \mathbf{f}_t(0) && (9)\\
&= \mathbf{A}(0)\mathbf{f}(0) + \mathbf{f}_t(0) && (10)\\
&= \mathbf{f}_t(0). && (11)
\end{aligned}$$

Hence, $\mathbf{f}(0)$ can non-trivially satisfy $\mathbf{A}(0)\mathbf{f}(0) = \mathbf{0}$ since $\det(\mathbf{A}) = \det(-\mathbf{A}^{\mathsf{T}}) = (-1)^3\det(\mathbf{A}) = -\det(\mathbf{A})$.

Axis of Rotation - Part II

If **A** is skew-symmetric and **f** is in its null-space then **A** and **f** must take the form,

$$\mathbf{A}(0) = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}, \quad \mathbf{f}(0) = \begin{bmatrix} -c \\ b \\ -a \end{bmatrix}. \tag{12}$$

Going back to our trace formula we have that,

$$\mathrm{tr}(\mathbf{U}(0)) = \mathrm{tr}(\mathbf{I}) = 3 \tag{13}$$

$$= 1 + 2\cos(\theta(0)) \Rightarrow \cos(\theta(0)) = 1, \tag{14}$$

which implies that $\theta(0) = \theta_0 = 0$.

Instantaneous Angular Velocity

We now have the instantaneous axis of rotation of the motion at $t = 0$. To find the instantaneous angular velocity, consider differentiating the trace formula,

$$\begin{aligned}\frac{d^2}{dt^2}\left[1 + 2\cos(\theta) - \text{tr}(\mathbf{U})\right]_{t=0} &= 2\left(-\ddot{\theta}_0 \sin(\theta_0) - \ddot{\theta}_0^2 \cos(\theta_0)\right) - \text{tr}(\mathbf{U}_{tt}(0)) \\ &= -2\dot{\theta}_0^2 - \text{tr}(\mathbf{U}_{tt}(0)) = 0,\end{aligned}$$

where $\mathbf{U}_{tt} = \mathbf{A}_t\mathbf{U} + \mathbf{A}\mathbf{U}_t = \mathbf{A}_t\mathbf{U} + \mathbf{A}\mathbf{A}\mathbf{U}$. At $t = 0$ we have,

$$\begin{aligned}\dot{\theta}_0^2 &= -\frac{1}{2}\text{tr}\left(\mathbf{U}_{tt}(0)\right) = -\frac{1}{2}\text{tr}\left(\mathbf{A}_t(0)\mathbf{U}(0) + \mathbf{A}^2(0)\mathbf{U}(0)\right) \\ &= -\frac{1}{2}\text{tr}\left(\mathbf{A}_t(0)\mathbf{I} + \mathbf{A}^2(0)\mathbf{I}\right) = -\frac{1}{2}\text{tr}\left(\mathbf{A}^2(0)\right) = a^2 + b^2 + c^2,\end{aligned}$$

which implies that $|\dot{\theta}_0| = |\mathbf{f}|$ and we say that $\mathbf{f}$ is the instantaneous angular velocity vector.

Summary - Part I

Given a rotating rigid body we would like to understand,

1. What is the instantaneous axis which the object is rotating about.
2. What is the instantaneous angular velocity of the object about this axis of rotation.

The answers to these questions are given, at $t = 0$, by,

1. The instantaneous axis of rotation is defined by the null-space of $\mathbf{A}(0)\mathbf{f}(0) = \mathbf{U}_t(0)\mathbf{U}(0)\mathbf{f}(0) = \mathbf{U}_t(0)\mathbf{f}(0) = \mathbf{0}$.
2. The instantaneous angular velocity can be found from the axis of rotation and trace formula,

$$|\dot{\theta}_t| = |\mathbf{f}| = -\frac{1}{2}\mathrm{tr}\left(\mathbf{A}^2\right) = -\frac{1}{2}\mathrm{tr}\left(\mathbf{U}_t^2\right) \tag{15}$$

Fluid Dynamics

The previous work can apply to bodies that deform when sheer stress is applied, which is the typical characterization of a fluid. Using this construct one can connect the terminology of fluid dynamics to concepts from vector calculus. We begin with the following:

- Position : Let $\mathbf{x} \in \mathbb{R}^3$ such that $\mathbf{x}(\mathbf{y}, 0) = \mathbf{y} \in \mathbb{R}^3$ be the position vector of the fluid at time t, which was located at $\mathbf{y}$ at time zero.
- Velocity : Define $\partial_t \mathbf{x} = \mathbf{x}_t(\mathbf{y}, t) = \mathbf{v}(\mathbf{y}, t)$.
- Locality : For fixed t the positions, $\mathbf{y}, \mathbf{x}$ should be connected through a linear map. This map is called the Jacobian,

$$\mathbf{J}(\mathbf{y}, t) = \frac{\partial x}{\partial y} \Rightarrow [\mathbf{J}]_{ij} = \frac{\partial x_i}{\partial y_j} \Rightarrow \mathbf{J}(\mathbf{y}, 0) = \mathbf{I} \tag{16}$$

Decomposition - I

Theorem (Lax : page 139-140) : Let $\mathbf{Z}$ be a mapping of Euclidean space into itself. Then $\mathbf{Z}$ has the decompositions:

- $\mathbf{Z} = \mathbf{RU}$ where $\mathbf{R}$ is a nonnegative self-adjoint mapping and $\mathbf{U}$ is unitary.
- $\mathbf{Z} = \mathbf{US}$ where $\mathbf{U}$ is a rotation and $\mathbf{S}$ is self-adjoint.

Taking the latter we have that the Jacobian has the representation, $\mathbf{J} = \mathbf{US}$. To understand the rate of rotation we consider,

$$\mathbf{J}_t = \mathbf{U}_t\mathbf{S} + \mathbf{U}\mathbf{S}_t = \mathbf{U}_t\mathbf{U}^{\mathsf{T}}\mathbf{J} + \mathbf{U}\mathbf{S}_t \tag{17}$$

$$= \mathbf{A}\mathbf{U}\mathbf{U}^{\mathsf{T}}\mathbf{J} + \mathbf{U}\mathbf{S}_t = \mathbf{A}\mathbf{J} + \mathbf{U}\mathbf{S}_t. \tag{18}$$

At $t = 0$ we have $\mathbf{J}_t(0) = \mathbf{A}(0)\mathbf{J}(0) + \mathbf{U}(0)\mathbf{S}_t(0) = \mathbf{A}(0) + \mathbf{S}_t(0)$.

Decomposition - II

We now have that at time $t = 0$ the time-derivative of the Jacobian has been decomposed into symmetric and anti-symmetric parts. We compare this against the definition of the Jacobian to get,

$$\frac{\partial}{\partial t}\mathbf{J} = \frac{\partial}{\partial t}\frac{\partial \mathbf{x}}{\partial \mathbf{y}} = \frac{\partial \mathbf{v}}{\partial \mathbf{y}} \Rightarrow [\mathbf{J}_t]_{ij} = \frac{\partial \mathbf{v}_i}{\partial \mathbf{y}_j}, \tag{19}$$

which has the decomposition,

$$[\mathbf{J}_t]_{ij} = \frac{1}{2}\left(\frac{\partial \mathbf{v}_i}{\partial \mathbf{y}_j} + \frac{\partial \mathbf{v}_j}{\partial \mathbf{y}_i}\right) + \frac{1}{2}\left(\frac{\partial \mathbf{v}_i}{\partial \mathbf{y}_j} - \frac{\partial \mathbf{v}_j}{\partial \mathbf{y}_i}\right) \tag{20}$$

$$= [\mathbf{S}_t(0)]_{ij} + [\mathbf{A}(0)]_{ij}\,. \tag{21}$$

Vorticity

Having **A** we now have that,

$$2a = \left(\frac{\partial \mathbf{v}_1}{\partial \mathbf{y}_2} - \frac{\partial \mathbf{v}_2}{\partial \mathbf{y}_1}\right),\ 2b = \left(\frac{\partial \mathbf{v}_1}{\partial \mathbf{y}_3} - \frac{\partial \mathbf{v}_3}{\partial \mathbf{y}_1}\right),\ 2c = \left(\frac{\partial \mathbf{v}_2}{\partial \mathbf{y}_3} - \frac{\partial \mathbf{v}_3}{\partial \mathbf{y}_2}\right), \tag{22}$$

which gives the instantaneous angular velocity vector as,

$$\mathbf{f}(0) = \begin{bmatrix} -c \\ b \\ -a \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{\partial \mathbf{v}_3}{\partial \mathbf{y}_2} - \frac{\partial \mathbf{v}_2}{\partial \mathbf{y}_3} \\ \frac{\partial \mathbf{v}_1}{\partial \mathbf{y}_3} - \frac{\partial \mathbf{v}_3}{\partial \mathbf{y}_1} \\ \frac{\partial \mathbf{v}_2}{\partial \mathbf{y}_1} - \frac{\partial \mathbf{v}_1}{\partial \mathbf{y}_2} \end{bmatrix} = \frac{1}{2}\text{curl } \mathbf{v} = \frac{1}{2}\nabla \times \mathbf{v}, \tag{23}$$

which is nothing more than one-half of the curl of the velocity field. This quantity is called the vorticity of the field.

Key Point: If the instantaneous angular velocity of the field is zero then the field has no curl.

Incompressibility

Theorem(Lax page 99) : Let $\mathbf{Y}(t)$ be a differentiable square matrix valued function. Then for those values of t for which $\mathbf{Y}(t)$ is invertible, $\frac{d}{dt}[\ln\det(\mathbf{Y})] = \mathrm{tr}(\mathbf{Y}^{-1}\mathbf{Y}_t)$. Thus,

$$\frac{d}{dt}[\ln\det(\mathbf{J})]_{t=0} = \frac{1}{\det(\mathbf{J}(0))}\frac{d}{dt}[\det(\mathbf{J})]_{t=0} = \frac{d}{dt}[\det(\mathbf{J})]_{t=0}$$

$$= \mathrm{tr}(\mathbf{J}^{-1}(0)\mathbf{J}_t(0) = \mathrm{tr}(\mathbf{J}_t(0)) = \sum_{i=1}^{3}\frac{\partial \mathbf{v}_i}{\partial \mathbf{y}_i} = \mathrm{div}(\mathbf{v}).$$

Key Point: Since the determinant of the Jacobian controls the change in volume as the fluid moves from **x** to **y** we have that the rate of change of this volume is equal to the divergence of the velocity field. Thus, a divergence free fluid is not undergoing volumatic changes and is called incompressible.

Summary - Part II

Using the concept of rotation to define the instantaneous axis of rotation and angular velocity, as seen through rigid bodies, we apply the same formalism to a fluid body. Doing so we find that,

- The instantaneous axis of rotation is given by the curl of the velocity field and conclude that an irrotational fluid is curl free. From calculus, we also call this sort of field conservative and known that a conservative field can be written as the gradient of a vector potential.
- This formalism can be used again to define the time rate of change of the volume of the fluid, which is then related to the divergence of the velocity field. From this we conclude that a divergence free field is incompressible.

Extra Credit

For twenty-five points of extra credit, applicable to your homework grade, please do the following:

1. Write up a two-paragraph summary of this lecture.
2. List three questions you have after this discussion.

Please submit your response by 5:00pm on Wednesday.

THANKS FOR LISTENING