Note Title 10/30/2006

10/30/06

Last time we proved the invertibility of the Fourier Transform, provided

$$K_\psi(x, x') = \frac{1}{2\pi}\int_{-\psi}^{\psi} e^{-ik(x'-x)}\, dx$$

$$= \frac{\sin \psi (x'-x)}{(x'-x)}$$

has the property that

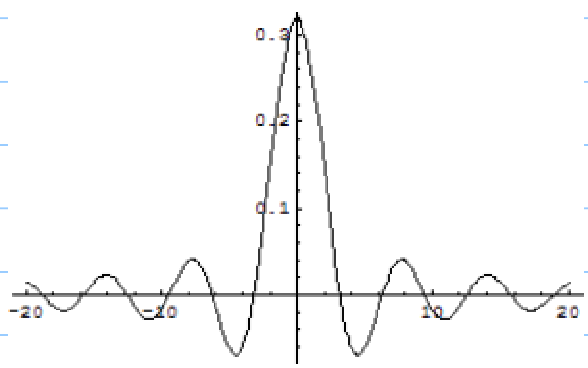
$$\lim_{\mu \to \infty} \int_{-\mu}^{\mu} f(x') K_\mu(x', x)\, dx' = f(x)$$

let $\lim_{\mu \to \infty} K_\mu(x', x) \equiv K(x', x)$

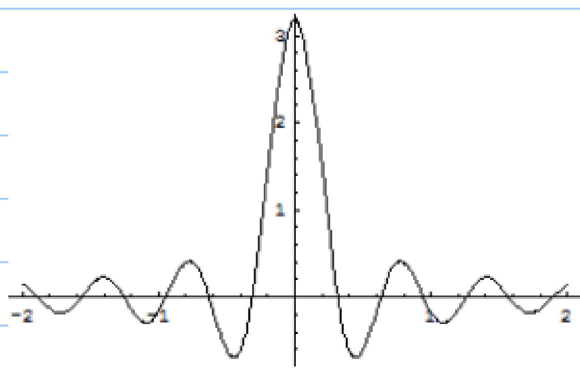
Notice that K depends only on $x' - x$, so we can write

$$\boxed{f(x) = \int_{-\infty}^{\infty} f(x') K(x' - x)\, dx'}$$

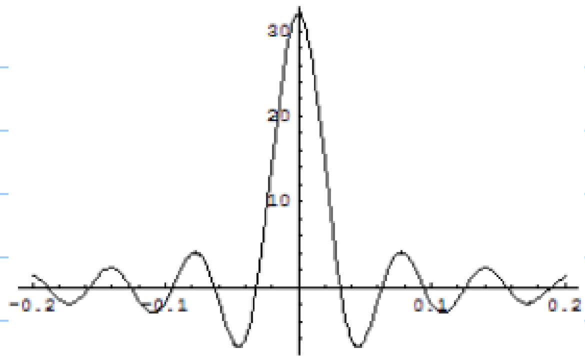
we "showed" graphically that as $\nu \to \infty$ k_ν converged to an infinite spike at the point $x' - x = 0$.



$\nu = 1 \quad [-20, 20]$



$\nu = 10 \quad [-2, 2]$



$\mu = 100 \quad [-.2, .2]$

$$\lim_{\mu \to \infty} K_\mu(x' - x) \equiv \delta(x' - x)$$

Let I be an interval

then

$$\int_I f(x)\,\delta(x-y)\,dx = \begin{cases} 0 & \text{if } y \notin I \\ f(y) & \text{if } y \in I \end{cases}$$

in particular

$$f(0) = \int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx$$

$$f(y) = \int_{-\infty}^{\infty} f(x)\, \delta(x-y)\, dx$$

The delta "function" is Not a regular function. It is only defined via its action on other "test function", which must be well-behaved.

E.g.

$\delta(-x) = ?$

$$\int_{-\infty}^{\infty} f(x)\, \delta(-x)\, dx$$

↑ test function

$x \to -x$

$$\int_{+\infty}^{-\infty} f(-x)\,\delta(x)\,d(-x)$$

$$= \int_{-\infty}^{\infty} f(-x)\,\delta(x)\,dx$$

$$= f(0)$$

So $\delta(x)$ is symmetric.

$\delta(-x) = \delta(x)$

$$\delta'(x) = ?$$

$$\int_{-\infty}^{\infty} f(x)\,\delta'(x)\,dx$$

$$= f(x)\,\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta(x)\,dx$$

$$= -f'(0)$$

$$\delta(ax) = ?$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(ax)\,dx$$

$$ax \to y$$

$$\frac{1}{a} \int_{?}^{?} f(y/a)\,\delta(y)\,dy$$

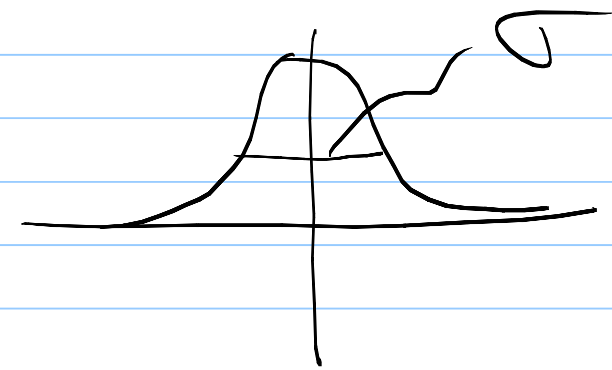
$$\frac{1}{|a|} \int_{-\infty}^{\infty} f(y/a)\,\delta(y)\,dy = \frac{1}{|a|} f(0)$$

So, we say

$$\delta(a x) = \frac{1}{|a|}\,\delta(x)$$

Fourièr transform of Gaussian

$$\frac{1}{\sigma\sqrt{2\pi}}\; e^{-x^2/2\sigma^2}$$



$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \, dx$$

$$= \frac{1}{\sigma 2\pi} \int_{-\infty}^{\infty} e^{-[x^2/2\sigma^2 + ikx]} \, dx$$

what to do?

complete the square

$$x^2/2\sigma^2 + ikx$$

$$= \frac{1}{2\sigma^2}\underbrace{\left[x^2 + ik2\sigma^2 x\right]}_{(x+ik\sigma^2)^2 - (ik\sigma^2)^2}$$

$$= \frac{1}{2\sigma^2}(x+ik\sigma^2)^2 + \frac{k^2\sigma^2}{2}$$

$$\frac{1}{\sigma 2\pi} e^{-k^2\sigma^2/2} \int_{-\infty}^{\infty} e^{-(x+ik\sigma^2)/2\sigma^2}\, dx$$

$$x + ik\sigma^2 = z$$

$$\int_{-\infty}^{\infty} e^{-z^2/2\sigma^2}\, dz$$

$$\vdots$$

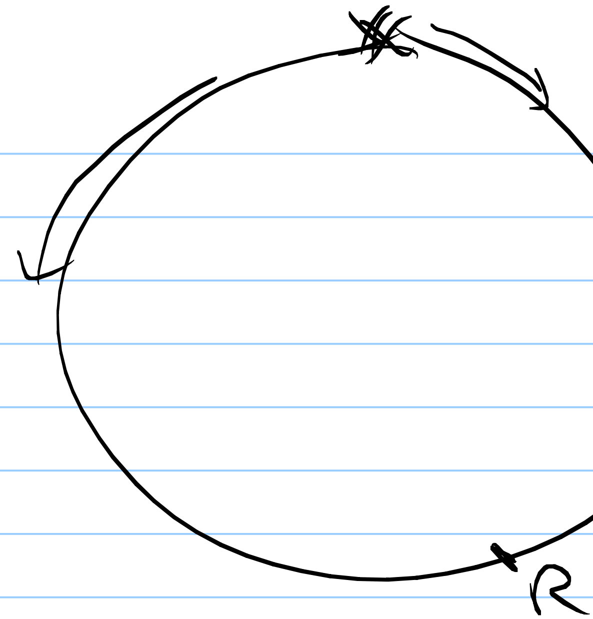
$$\int_{-\infty}^{\infty} e^{-x^2}\, dx$$

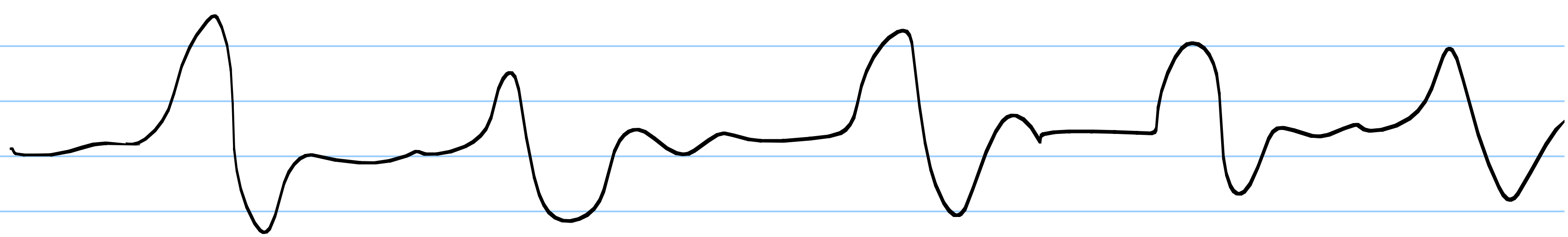
You can fill in the details

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \Rightarrow \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2\sigma^2}$$

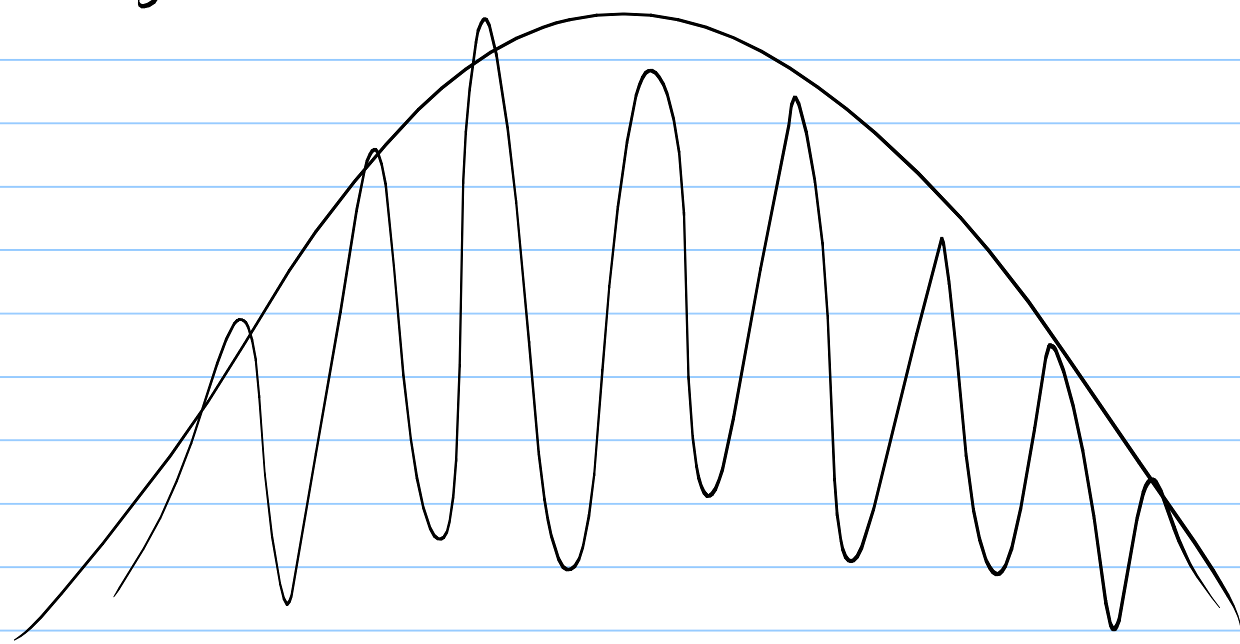
$$\frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma^2} \Rightarrow \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\omega^2\sigma^2}$$

Surface waves





FT of a long train of pulses



FT of a short
Train of pulses