MATH348-Advanced Engineering Mathematics	Homework: One

Vector Spaces, ODEs and Geometry

Text: Chapter 5 Lecture Notes: N/A Slides: N/A

Quote of Homework One	
Let's start at the very beginning, a very good place to start.	
	Ernest Lehman : The Sound of Music (1965)

1. Second Order Linear ODEs with Constant Coefficients

Often it is the case that ODEs appear in the form,

$$ay'' + by' + cy = f(x), \quad a, b, c \in \mathbb{R}. \tag{1}$$

1.1. **Homogeneous Solution.** Solve the associated homogeneous problem.

1.2. **Resonant Solutions.** A interesting prediction of the mathematics is resonant harmonic motion, which is an oscillatory solution whose amplitude grow in time. [1] Find the general solution the previous ODE when $a = c = 1$, $b = 0$ and $f(x) = \cos(x)$.[2]

2. Power Series and Hyperbolic Trigonometric functions

Consider the ordinary differential equation:

$$y'' - y = 0 \tag{2}$$

2.1. **General Solution - Exponential Form.** Show that the solution to (2) is given by $y(x) = c_1e^x + c_2e^{-x}$.

2.2. **General Solution -Hyperbolic Form.** Show that $y(x) = b_1 \sinh(x) + b_2 \cosh(x)$ is a solution to (2) where $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ and $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$.

2.3. **Conversion from Standard to Nonstandard Form.** Show that if $c_1 = \dfrac{b_1 + b_2}{2}$ and $c_2 = \dfrac{b_1 - b_2}{2}$ then $y(x) = c_1e^x + c_2e^{-x} = b_1 \cosh(x) + b_2 \sinh(x)$.

2.4. **Relation to Power-Series.** Assume that $y(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ to find the general solution of (2) in terms of the hyperbolic sine and cosine functions. [3]

3. 2^{nd} Order Linear ODE: General Results

Typically, one arrives at the second-order linear ODE,

$$a(x)y'' + b(x)y' + c(x)y = f(x), \tag{4}$$

from Newton's or Kirchoff's law.

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[1] These solutions are interesting in the sense that a bounded external force produces an unbounded solution. This has to do with the external force pumping energy into the system in 'just the right way.'

[2] Though you could use the previous formulae it is more efficient and a good review to do this problem via undetermined coefficients.

[3] The hyperbolic sine and cosine have the following Taylor's series representations centred about $x = 0$,

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}. \tag{3}$$

It is worth noting that these are basically the same Taylor series as cosine/sine with the exception that the signs of the terms do not alternate. From this we can gather a final connection for wrapping all of these functions together. If you have the Taylor series for the exponential function and extract the even terms from it then you have the hyperbolic cosine function. Taking the hyperbolic cosine function and alternating the sign of its terms gives you the cosine function. Extracting the odd terms from the exponential function gives the same statements for the hyperbolic sine and sine functions. The reason these functions are connected via the imaginary number system is because when i is raised to integer powers it will produce these exact sign alternations. So, if you remember $e^x = \sum_{n=0}^{\infty} x^n/n!$ and $i = \sqrt{-1}$ then the rest (hyperbolic and non-hyperbolic trigonometric functions) follows!

3.1. **Second Linearly Independent Solution.** Suppose that $a(x) = 1,\ b(x) = 4,\ c(x) = 4,\ f(x) = e^{-2x}$. [4] We know a solution to this problem is $y_1(x) = e^{-2x}$. Using the formula,

$$y_2(x) = k(x)y_1(x),\ k(x) = \int \frac{p(x)}{[y_1(x)]^2}dx,\ p(x) = e^{-\int (b(x)/a(x))dx}, \tag{5}$$

find a second linearly independent solution to the ODE.

3.2. **Particular Solution: Part I.** Using the formula,

$$y_p(x) = y_2 \int \frac{f(x)y_1(x)}{a(x)W(x)}dx - y_1 \int \frac{f(x)y_2(x)}{a(x)W(x)}dx,\ W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x), \tag{6}$$

find the form for the particular solution. [5]

3.3. **Particular Solution: Part II.** With our newfound trust, we use the previous formula on a problem that couldn't have been analyzed through previous methods. Solve the previous ODE where $a(x) = 1,\ b(x) = 0,\ c(x) = 1,\ f(x) = \sec(x)$, where $x > 0$.

4. Abstract Vector Spaces - Function Spaces

Given,

$$\left[m\frac{d^2}{dt^2} + k\right] y = 0,\ m, k \in \mathbb{R}^+ \tag{7}$$

$$\left[\frac{d}{dt} - \mathbf{A}\right] \mathbf{Y} = 0,\ \mathbf{A} \in \mathbb{R}^{2\times 2} \tag{8}$$

4.1. **Equivalence of Equations.** Find the change of variables that maps (7) onto (8) and using this define $\mathbf{Y}$ and $\mathbf{A}$.

4.2. **Function Spaces.** Find the general solution to (8) and for $m = k = 1$ sketch its associated real phase-portrait.

5. Orthogonal Expansions

Given,

$$\hat{\mathbf{i}} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \qquad \hat{\mathbf{j}} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}. \tag{9}$$

5.1. **Orthonormality - Part I.** Show that the vectors are orthonormal by verifying the inner-products $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0$ and $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = 1$.

5.2. **Orthogonal Representation I.** Show that any vector for $\mathbb{R}^2$ can be created as a linear combination of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$. That is, given,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}}, \tag{10}$$

show that c_1, c_2, can be found in terms of x_1 and x_2.

5.3. **Orthonormality - Part II.** Show that $\langle f, g\rangle = (f, g) = \int_{-\pi}^{\pi} f(x)g(x)\,dx$ satisfies the three axioms of a *Real Inner Product Space* and that $\langle \cos(nx), \cos(mx)\rangle = \langle \sin(nx), \sin(mx)\rangle = \pi\delta_{nm},\ \langle \cos(nx), \sin(mx)\rangle = 0$ for all $n, m \in \mathbb{N}$.

5.4. **Orthogonal Representation II.** Show that if $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$ then

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \tag{11}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \tag{12}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx. \tag{13}$$

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[4] This problem is degenerate in the sense that it contains a repeated eigenvalue. Worse, the inhomogeneous term competes with the associated eigenfunction. You can solve this completely using techniques from your previous course work. We will use some formula to justify these techniques.

[5] You might notice that this can be done via the method of undetermined coefficients, which is considerably easier even if you have to multiply your 'guess' by two factors of x!