Delta functions and point sources

We've already seen indications that point charges behave a bit wonky, even though they do seem to exist experimentally. I'm afraid this is going to get a bit worse before it gets better.

Let's look at Gauss's law again. If we take the divergence of an E-field, we should recover the charge density that produced that field:

$$\nabla \cdot \vec{E} = \rho / \varepsilon_0$$

What happens if we take the divergence of the field made by a point charge at the origin?

$$\nabla \cdot \left(\frac{q}{4\pi\varepsilon_0} \frac{\hat{r}}{r^2} \right) = ?$$

Apply the divergence operator in spherical coordinates

$$= \frac{1}{r^2} \frac{d}{dr} \left(\cancel{r^2} \cdot \frac{q}{4\pi\varepsilon_0 \cancel{r^2}} \right)$$

$$= \frac{1}{r^2} \frac{d}{dr} \left(\frac{q}{4\pi\varepsilon_0} \right) = 0 \; ?$$

Well, according to that, the divergence of something like $\frac{\hat{r}}{r^2}$ is zero everywhere. Thus, so must ρ be zero everywhere. But that can't be right. So what's the catch?

The catch is that the operator $\frac{1}{r^2} \frac{d}{dr} \left(r^2 \right)$ isn't super well-defined at the origin. All we can conclude from the above is that $\nabla \cdot \vec{E}$ is zero everywhere <u>but</u> the origin.

To deal with the origin, let's take a look at delta functions first.

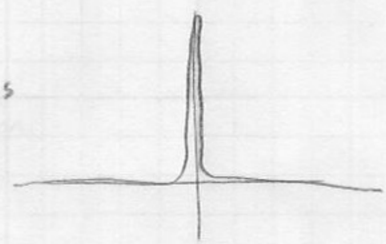
A delta function is used to represent a finite amount of stuff compressed into an essentially zero-dimensional domain. In one dimension, a delta function $\delta(x)$ is *defined* as the thingy that satisfies these two properties:

$\delta(x)$ is a delta function if:

1) $\delta(x) = 0$ $\quad \forall x$ except $x=0$, where δ is undefined

2) $\int \delta(x-a) f(x) dx = f(a)$ if a is in the domain of integration, or zero otherwise

Basically, its a very sharply peaked function that, when present in an integral, plucks out the value of another function at one point.



So how do we represent $\rho(\vec{x})$ for a point charge of size q in 3D? How about a delta function?

$$\rho(\vec{x}) = q\delta^3(\vec{x})$$

(Or, if the point is at $\vec{x}'$ instead of the origin, $q\delta^3(\vec{x}-\vec{x}')$)

With that in mind, since $\nabla \cdot \vec{E} = \rho/\varepsilon_0$, it should be the case that

$$\nabla \cdot \left(\frac{q}{4\pi\varepsilon_0} \frac{\hat{r}}{r^2} \right) = q\delta^3(\vec{x})$$

And therefore $\qquad \nabla \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{x})$

Is that true?

Well, its true if

1) $\nabla \cdot \frac{\hat{r}}{r^2} = 0$ everywhere but the origin. And we established that.

2) $\int \left(\nabla \cdot \frac{\hat{r}}{r^2} \right) d^3x = 4\pi$ for a domain including the origin.

Let's integrate $\nabla \cdot \frac{\hat{r}}{r^2}$ over a sphere of radius R.

Just doing that as a volume integral is tricky since the integrand is undefined at the origin, so dodge the bad part by using the divergence theorem:

$$\int \left(\nabla \cdot \frac{\hat{r}}{r^2} \right) d^3x = \oint \frac{\hat{r}}{r^2} \cdot d\vec{A}$$

$$= \oint \frac{\hat{r}}{r^2} \cdot r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$= \oint \sin\theta \, d\theta \, d\phi \quad \text{(over a sphere)}$$

$$= 4\pi$$

So it checks out. $\nabla \cdot \frac{\hat{r}}{r^2} = 4\pi \delta^3(r)$, and thus

$$\nabla \cdot \left(\frac{q}{4\pi\varepsilon_0} \frac{\hat{r}}{r^2} \right) = q\delta^3(r)/\varepsilon_0 = \rho/\varepsilon_0$$