Reading: Today: G 9.2

Tomorrow: G 9.3

1-D Wave equation: $\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$

3-D Wave equation: $\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$

General solution:

$$\tilde{f} = \tilde{A} e^{i(\vec{k}\cdot\vec{r} - \omega t)} = \tilde{A} e^{i(k_x x + k_y y + k_z z - \omega t)}$$

points in direction of propagation.

$$v = \frac{\omega}{|\vec{k}|}$$

In 3-D:

Any function can be written as:

$$\tilde{f}(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{A}(k_x,k_y,k_z)\, e^{i(\vec{k}\cdot\vec{r})}\, dk_x\, dk_y\, dk_z$$

for waves replace $\vec{k}\cdot\vec{r} \Rightarrow \vec{k}\cdot\vec{r} - \omega t$

EM Waves

Show using ME: $\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$; $\nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$

assuming $\rho, \vec{J} = \phi$

Hint: Take some curls of curls.

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

This shows $v_{\text{light, radio, whatever EM}} = \frac{1}{\sqrt{\epsilon_0\mu_0}} = c$

These eqns say:

$$\vec{E}_x = \tilde{E}_{ox}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\vec{E}_y = \tilde{E}_{oy}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\vec{E}_z = \tilde{E}_{oz}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Same for $\vec{B}$

$\nabla\cdot\vec{E} = \phi \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \phi$

Let's pick a coordinate system where $\vec{k} = k\hat{z}$

$$\vec{k}\cdot\vec{r} = kz$$

$$\frac{\partial E_x}{\partial x} = \phi,\quad \frac{\partial E_y}{\partial y} = \phi,\quad \frac{\partial E_z}{\partial z} = ik\tilde{E}_{oz}e^{i(kz-\omega t)}$$

$\Rightarrow$ $\vec{E}$ for a wave has to be perpendicular to $\vec{k}$

$\Rightarrow$ Same has to go for $\vec{B}$.

Hint to this is the Poynting vector:

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}\left(\vec{B}_0 e^{i(kz-\omega t)}\right)$$

$$\vec{\nabla}\times\vec{E} = +i\omega\vec{B} = i\omega\left(B_x\hat{x} + B_y\hat{y}\right)$$

$$\left(\cancelto{0}{\frac{\partial E_y}{\partial x}} - \cancelto{0}{\frac{\partial E_x}{\partial y}}\right)\hat{z} + \left(\cancelto{0}{\frac{\partial E_z}{\partial y}} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \cancelto{0}{\frac{\partial E_z}{\partial x}}\right)\hat{y}$$

$$-\frac{\partial E_y}{\partial z} = i\omega B_x = -ik\tilde{E}_y$$

$$\frac{\partial E_x}{\partial z} = i\omega B_y = ik\tilde{E}_x$$

$$\Rightarrow \quad \omega B_x = -k\tilde{E}_y, \quad \omega B_y = k\tilde{E}_x$$

$$\vec{B} = \frac{1}{\omega}\left(-k\tilde{E}_y\hat{x} + k\tilde{E}_x\hat{y}\right)$$

$$= \frac{1}{\omega}\left(\vec{k}\times\vec{E}\right) = \frac{1}{\omega}\left(k\hat{z}\times\left(\tilde{E}_x\hat{x} + \tilde{E}_y\hat{y}\right)\right)$$

$\vec{B}$ vector is also perpendicular to $\vec{E}$ for a plane wave.

$\Rightarrow$ The general soln for Maxwell's w/ no sources at angular frequency ω is

$$\vec{E} = \tilde{E}_0\hat{n}\left(e^{i(\vec{k}\cdot r - \omega t)}\right); \quad \underline{\hat{n}\cdot\vec{k} = 0}.$$

$$\vec{B} = \frac{1}{\omega}\left(\vec{k}\times\vec{E}\right) = \frac{1}{c}\left(\hat{k}\times\vec{E}\right) \qquad c = \frac{|\vec{k}|}{\omega}$$

You could write $\vec{E}$ as

$$\vec{E} = \left(\vec{C}\times\hat{k}\right)e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Time averages

In terms of our complex fields:

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \vec{E} = \mathrm{Re}(\tilde{\vec{E}})$$

What does $\tilde{\vec{E}} \cdot \tilde{\vec{E}}^* = ?$ in terms of the real amplitude E_0.

$$\text{Real } \vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$$

$$\tilde{\vec{E}} \cdot \tilde{\vec{E}}^* = E_0^2$$

$$E^2 = E_0^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t)$$

$$\langle E^2 \rangle = \frac{1}{2} E_0^2$$

$$\Rightarrow \tilde{\vec{E}} \cdot \tilde{\vec{E}}^* = 2\langle E^2 \rangle$$

$$\langle E^2 \rangle = \frac{1}{2}(\tilde{\vec{E}} \cdot \tilde{\vec{E}}^*)$$

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0} \tilde{\vec{E}} \times \tilde{\vec{B}}^*$$

$$\langle u_{em} \rangle = \frac{1}{2}\left(\frac{1}{2\epsilon_0} \tilde{\vec{E}} \cdot \tilde{\vec{E}}^* + \frac{\mu_0}{2} \tilde{\vec{B}} \cdot \tilde{\vec{B}}^* \right)$$

Derive Maxwell's eqns in Gaussian units:

Define $\vec{e} = \sqrt{\epsilon_0}\,\vec{E}$, $\vec{b} = \frac{1}{\sqrt{\mu_0}}\vec{B}$

$$\rho' = \frac{\rho}{4\pi\sqrt{\epsilon_0}} \qquad \vec{J}' = \frac{\vec{J}}{4\pi\sqrt{\epsilon_0}}$$

Rewrite ME in terms of new vars:

$$\vec{\nabla}\cdot\vec{e} = 4\pi\rho' \qquad \vec{\nabla}\times\vec{e} = -\frac{1}{c}\frac{\partial\vec{b}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{b} = \emptyset \qquad \vec{\nabla}\times\vec{b} = \frac{1}{c}\frac{\partial\vec{e}}{\partial t} + \frac{1}{c}4\pi\vec{J}'$$

Let's do something worse

G 7.60: If magnetic charge,

$$\vec{E}' = \vec{E}\cos\alpha + c\vec{B}\sin\alpha$$

$$\vec{B}' = \vec{B}\cos\alpha - \frac{1}{c}\vec{E}\sin\alpha$$

$$\left.\begin{aligned} \rho_e' &= \rho_e\cos\alpha + \frac{\rho_m}{c}\sin\alpha \\ \rho_m' &= \rho_m\cos\alpha - c\rho_e\sin\alpha \end{aligned}\right\} \text{same for } J.$$

$\alpha = 45^\circ$

$$\vec{E}' = \frac{1}{\sqrt{2}}\vec{E} + \frac{c}{\sqrt{2}}\vec{B} \rightarrow \boxed{\vec{B} = \frac{1}{\sqrt{2}}\vec{B}' + \frac{1}{c\sqrt{2}}\vec{E}'}$$

$$\vec{B}' = \frac{1}{\sqrt{2}}\vec{B} - \frac{1}{\sqrt{2}c}\vec{E} \rightarrow \boxed{\vec{E} = \frac{1}{\sqrt{2}}\vec{E}' - \frac{c}{\sqrt{2}}\vec{B}'}$$

$$\boxed{\rho_e' = \rho\frac{1}{\sqrt{2}}} \; ; \; \boxed{\rho_m' = -c\rho\frac{1}{\sqrt{2}}}$$

$$\vec{\nabla}\cdot\vec{E}' = \frac{1}{\sqrt{2}}\vec{\nabla}\cdot\vec{E} + \frac{c}{\sqrt{2}}\cancelto{0}{\vec{\nabla}\cdot\vec{B}} = \frac{1}{\sqrt{2}}\frac{\rho}{\epsilon_0} = \rho_e'/\epsilon_0$$

$$\vec{\nabla}\cdot\vec{B}' = \frac{1}{\sqrt{2}}\cancelto{0}{\vec{\nabla}\cdot\vec{B}} - \frac{1}{\sqrt{2}c}\vec{\nabla}\cdot\vec{E} = -\frac{1}{\sqrt{2}c}\frac{\rho}{\epsilon_0} = +\frac{1}{c^2\epsilon_0}\rho_m'$$

$$\boxed{\vec{\nabla}\cdot\vec{B}' = \mu_0\rho_m' \quad , \quad \vec{\nabla}\cdot\vec{E} = \rho_e'/\epsilon_0}$$

$$\vec{\nabla}\times\vec{E}' = \frac{1}{2}\vec{\nabla}\times\vec{E} + \frac{c}{\sqrt{2}}\vec{\nabla}\times\vec{B}$$

$$= \frac{1}{\sqrt{2}}\left(-\frac{\partial\vec{B}}{\partial t}\right) + \frac{c}{\sqrt{2}}\left(\mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$= \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{2}}\vec{B}' + \cancelto{0}{\frac{1}{c\sqrt{2}}\vec{E}'}\right) + \frac{c}{\sqrt{2}}\mu_0\vec{J}\right.$$
$$\left. + \epsilon_0\mu_0 c\frac{\partial}{\partial t}\left(\cancelto{0}{\frac{1}{\sqrt{2}}\vec{E}'} - \frac{c}{\sqrt{2}}\vec{B}'\right)\right)$$

$$= \frac{c}{\sqrt{2}}\mu_0\vec{J} - \frac{1}{2}\frac{\partial}{\partial t}\vec{B}' - \frac{1}{2}\frac{\partial}{\partial t}\vec{B}'$$

$$= \frac{\mu_0}{\sqrt{2\epsilon_0\mu_0}}\vec{J}' - \frac{\partial\vec{B}}{\partial t} = -\mu_0\vec{J}_m' - \frac{\partial\vec{B}}{\partial t}$$

You get $\vec{F} = q_e\vec{E} + q_m\vec{B} + q_e\vec{v}_e\times\vec{B} - q_m\vec{v}_m\times\vec{E}$

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