Homework #1 Solution:

1. We have the following augmented matrix representation of (1) and (2),

$$\left[\begin{array}{cc|c} a & b & f \\ c & d & g \end{array}\right] \sim^{R1=dR1-bR2}_{R1=aR2-cR1} \left[\begin{array}{cc|c} ad-cb & bd-db & df-bg \\ ac-ca & ad-cb & dg-cf \end{array}\right] =$$

$$=^{(\star)} \left[\begin{array}{cc|c} ad-cb & 0 & df-bg \\ 0 & ad-cb & ag-cf \end{array}\right] \sim^{R2=R2/(ad-cb)}_{R1=R1/(ad-cb)} \left[\begin{array}{cc|c} 1 & 0 & \frac{df-bg}{ad-cb} \\ 0 & 1 & \frac{ag-cf}{ad-cb} \end{array}\right]$$

which is equivalent to the linear system,

$$(1') \qquad 1 \cdot x_1 + 0 \cdot x_2 = x_1 = fracdf - bgad - cb$$

$$(2') \qquad 0 \cdot x_1 + 1 \cdot x_2 = \frac{ag-cf}{ad-cb}$$

Note, to do the division at $(\star)$ we have assumed that

$$ad - bc \neq 0$$

This is a common statement which places restriction on a,b,c,d.

2. We begin by writing the 3 linear equations as the augmented matrix,

$$\left[\begin{array}{ccc|c} 6 & 18 & 1 & 20 \\ -1 & -3 & 8 & 4 \\ 5 & 15 & -9 & 11 \end{array}\right] \begin{array}{c} R1 \to R3 \\ R3 \to R2 \\ R2 \to R1 \end{array} \sim \left[\begin{array}{ccc|c} -1 & -3 & 8 & 4 \\ 5 & 15 & -9 & 11 \\ 6 & 18 & -4 & 20 \end{array}\right] \sim$$

$$\sim_{\substack{R2=5R1+R2 \\ R3=6R1+R3}} \left[\begin{array}{ccc|c} -1 & -3 & 8 & 4 \\ 0 & 0 & 40-9 & 11+20 \\ 0 & 0 & 44 & 44 \end{array}\right] \sim^{R3=R3/44}_{R2=R2/44}$$

$$\sim_{R1=-R1} \left[\begin{array}{ccc|c} 1 & 3 & -8 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim^{R3=R3+R2} \left[\begin{array}{ccc|c} 1 & 3 & 8 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\sim_{R1=R1+8R2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Which corresponds to the row equivalent linear system,

$$x_1 + 3x_2 = 4$$

$$x_3 = 1$$

Letting $x_2 = t$ implies that the general solution set is given by,

$$(\star) = \begin{array}{l} x_1 = -3t + 4 \\ x_2 = t \\ x_3 = 1 \end{array} , \qquad t \in \Re$$

With $x_1$ dependent on the <u>one</u> free variable $x_2$
$(\star)$ parameterizes a 2-D line in 3-D space

3. We have the polynomial

$$p(t) = a_0 + a_1 t + a_2 t^2$$

and the data points $(1, 12), (2, 15), (3, 16)^{(\star)}$. This generates 3 linear equations

$$\begin{aligned} p(1) &= a_0 + a_1 + a_2 = 12 \\ p(2) &= a_0 + 2a_1 + 4a_2 = 15 \\ p(3) &= a_0 + 3a_1 + 9a_2 = 16 \end{aligned}$$

and the corresponding augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 16 \end{array}\right] \sim_{\substack{R3=R3-R1 \\ R2=R2-R1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 8 & 4 \end{array}\right] \sim$$

$$\sim_{R3=R3-2R2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & -2 \end{array}\right] \sim_{R3=R3/2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \sim$$

$$\sim_{\substack{R1=R1-R3 \\ R2=R2-3R3}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 13 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right] \sim_{R1=R1-R2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right] \sim$$

The row equivalent linear system is then,

$$\begin{aligned} a_0 &= 7 \\ a_1 &= 6 \\ a_2 &= -1 \end{aligned}$$

which implies that $p(t) = 7 + 6t - t^2$ is the quadratic polynomial which indicates $(\star)$.

4.

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array}\right] \sim_{R3=R3-3R1} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{array}\right]$$

corresponds to the linear system

$$\begin{aligned} x_1 + 3x_2 &= 2 \\ (h-9)x_2 &= k-6 \end{aligned}$$

a) For this system to be consistent with a unique solution,

$$(h-9)x_2 = k-6 \Rightarrow x_2 = \frac{k-6}{h-9}, \text{ assuming } h-9 \neq 0$$

thus $h - 9 \neq 0 \Rightarrow h \neq 9$ will yield no free variables and the linear system is consistent with a unique point of intersection of the two lines.

b) For infinitely many solutions (for $x_2$ to be a free variable) we require that

$$(h-9)x_2 = k - 6 \Leftrightarrow 0 \cdot x_2 = 0 \Rightarrow h = 9, k = 6$$

Thus $x_2$ is free.

c. For no solutions we require,

$$(h-9)x_2 = k - 6 \Leftrightarrow 0 \cdot x_2 = c, c \in \Re, c \neq 0$$

This implies that $h = 9$ and $k \neq 6$. Thus the augmented column is a pivot column and the system has no solutions.

5. If $\vec{b} = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix}$ is a linear combination of the vectors, $\vec{a}_1 = \begin{bmatrix} 5 \\ -4 \\ 9 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 3 \\ 7 \\ -2 \end{bmatrix}$ formed from the columns of $A = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix}$ then there must exist $x_1, x_2 \in \Re$ such that

$$\vec{a}_1 \cdot x_1 + \vec{a}_2 \cdot x_2 = \vec{b} \Leftrightarrow \begin{array}{l} 5x_1 + 3x_2 = 22 \\ -4x_1 + 7x_2 = 20 \\ 9x_1 - 2x_2 = 15 \end{array}$$

To determine if this is true we row reduce the augmented matrix,

$$\left[\begin{array}{cc|c} 5 & 3 & 22 \\ -4 & 7 & 20 \\ 9 & -2 & 15 \end{array}\right] \sim_{\substack{R3=5R3-9R1 \\ R2=5R2+4R1}} \left[\begin{array}{cc|c} 5 & 3 & 22 \\ 0 & 47 & 188 \\ 0 & -37 & -123 \end{array}\right] \sim$$

$$\sim_{R3=47R3+37R2} \left[\begin{array}{cc|c} 5 & 3 & 22 \\ 0 & 47 & 188 \\ 0 & 0 & 1175 \end{array}\right]$$

which corresponds to the linear system,

$$\begin{array}{ll} 5x_1 + 3x_2 = 22 & \\ 47x_2 = 188 & \\ 0 \cdot x_2 = 1175 & (\star) \end{array}$$

There is no $x_2$ such that $(\star)$ can be satisfied. Thus, the linear system is inconsistent and $\vec{b}$ is <u>not</u> a linear combination of $\vec{a}_1$ and $\vec{a}_2$.