Calculating velocities:

4-vector is invariant under Lorentz transform.

$\vec{\mathbb{X}} \equiv (\vec{x}, ict)$

$d\mathbb{X} = (d\vec{x}, ic\,dt)$ both invariant

$ds = \sqrt{dx_\mu dx_\mu}$ = invariant,

so $d\tau = \frac{ds}{-ic}$ is also. = "proper time"

$\therefore$ define $\vec{U} = d\vec{\mathbb{X}}/d\tau$ 4-vector velocity.

$$= \left(\frac{d\vec{x}}{d\tau}, ic\frac{dt}{d\tau} \right)$$

"ordinary velocity" $\vec{u} = \frac{dx_j}{dt}$

$$d\tau = \frac{i}{c}\sqrt{dx_j dx_j - c^2 dt^2}$$

$$= dt\sqrt{1 - \frac{dx_j dx_j}{c^2 dt^2}} = dt\sqrt{1 - u^2/c^2}$$

$\therefore \vec{U} = (\vec{u}\gamma_u, ic\gamma_u)$ w/ $\gamma_u = (1 - u^2/c^2)^{-1/2}$

> 1

4-vector mom. $\vec{P} \equiv m_0 \vec{U}$ m_0 = rest mass

$$= \left(m_0\gamma_u\vec{u}, m_0\gamma_u \cdot ic \right)$$

define effective mass

$$m' = m_0\gamma_u$$

Example: motion under constant force

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad \text{w/ } p=0 \text{ at } t=0$$

$$p(t) = Ft = \frac{m_0 u}{\sqrt{1-u^2/c^2}} \qquad \frac{(Ft)^2}{m_0 c^2} = \frac{u/c}{1-u^2/c^2}$$

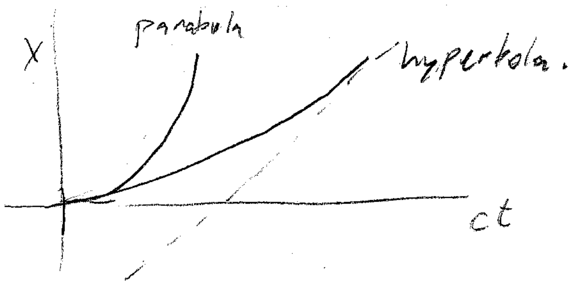
solve for $u \rightarrow \dfrac{(F/m_0)t}{\sqrt{1+(Ft'/m_0c^2)^2}}$ numerator is classical result.

integrate to get $x(t)$:

$$x(t) = \frac{F}{m_0}\int_0^t \frac{t'\,dt'}{\sqrt{1+(Ft'/m_0c^2)^2}}$$

$$= \frac{m_0c^2}{F}\sqrt{1+(Ft'/m_0c^2)^2}\,\Big|_0^t$$

$$= \frac{m_0c^2}{F}\left(\sqrt{} - 1\right)$$



energy in relativistic motion

3-D force $\vec{F} = \frac{d}{dt}\left(\frac{m_0 \vec{u}}{\sqrt{1-u^2/c^2}}\right)$

calc work done on particle

$$\vec{F}\cdot\vec{u} = \frac{dT}{dt} = \vec{u}\cdot\frac{d}{dt}\frac{m_0\vec{u}}{\sqrt{1-u^2/c^2}}$$

can show this can be written as

$$\frac{d}{dt}\left(\frac{m_0 c^2}{\sqrt{1-u^2/c^2}}\right)$$ (see next page)

But total energy is $m_0c^2/\sqrt{1-u^2/c^2}$

if particle is at rest at $t=0$

total energy is

$$W = T + m_0c^2 = mc^2$$

↳ rel. mass

Connect total energy to ordinary momentum

note $\vec{\mathbb{P}}\cdot\vec{\mathbb{P}} = \frac{m_0^2u^2}{(1-u^2/c^2)} - \frac{m_0^2c^2}{(1-u^2/c^2)} = -m_0^2c^2\frac{\cancel{\left(-\frac{u^2}{c^2}+1\right)}}{\cancel{(1-u^2/c^2)}}$

$$= p^2 - m_R^2c^2 = -m_0^2c^2$$

since $W = m_Rc^2$

$$\underline{W^2 = m_R^2c^4 = p^2c^2 + m_0^2c^4}$$

useful in kinematics

for massless photon $m_0 \to 0$ $W = pc = \hbar k\cdot c = \hbar\omega$ ✓

$$\left(\frac{m_0 \vec{u}}{\sqrt{1-u^2/c^2}}\right)' \cdot \vec{u} = \frac{m_0 \vec{u}' \cdot u}{\sqrt{}} + \frac{m_0 \vec{u}\cdot\vec{u}}{()^{3/2}} \left(-\frac{1}{2}\right)\left(-\frac{2}{c^2}\vec{u}\cdot\vec{u}'\right)$$

$$= \frac{m_0 \vec{u}' \cdot \vec{u}}{\sqrt{1-u^2/c^2}} \left(1 + \frac{u^2/c^2}{1-u^2/c^2}\right)$$

$$= \frac{m_0 \vec{u}' \cdot \vec{u}}{(1-u^2/c^2)^{3/2}} \left(1 \cancel{- \frac{u^2}{c^2} + u^2/c^2}\right)$$

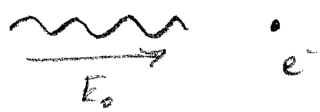
$$1$$

compare to

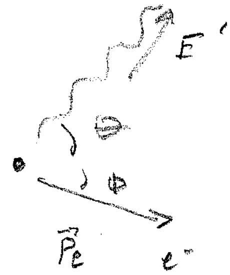
$$\frac{d}{dt}\left(\frac{m_0 c^2}{\sqrt{1-u^2/c^2}}\right) = m_0 c^2 \frac{\left(-\frac{1}{2}\right)\left(-\frac{2}{c^2}(\vec{u}' \cdot \vec{u})\right)}{(1-u^2/c^2)^{3/2}} \quad \checkmark$$

Example - compton scattering

photon collides w/ free electron



$\Longrightarrow$



$p_{in} = E_0/c$

cons momentum in vertical direction

$$p_e \sin\phi = \frac{E_0'}{c}\sin\theta$$

horiz.

$$\frac{E_0}{c} = \frac{E_0'}{c}\cos\theta + p_e\cos\phi = \frac{E_0'}{c}\cos\theta + p_e\sqrt{1-\left(\frac{E_0'\sin\theta}{c\,p_e}\right)^2}$$

$$p_e^2c^2 = (E_0 - E_0'\cos\theta)^2 + (E_0'\sin\theta)^2$$
$$= E_0^2 + E_0'^2 - 2E_0E_0'\cos\theta$$

Finally cons. energy :

$$E_0 + m_0c^2 = E_0' + \underbrace{\sqrt{m_0^2c^4 + p_e^2c^2}}_{E_e}$$
$$= E_0' + \sqrt{m_0^2c^4 + E_0^2 + E_0'^2 - 2E_0E_0'\cos\theta}$$

find E_0':

$$\rightarrow E_0' = \frac{1}{(1-\cos\theta)/m_0c^2 + 1/E_0}$$

let $E_0 = hc/\lambda$

$$\frac{\lambda'}{hc} = \frac{1-\cos\theta}{m_0c^2} + \frac{\lambda}{hc}$$

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_0c}(1-\cos\theta) \quad \text{compton shift}$$

$$h/m_0c \sim \frac{2\pi\times10^{-34}}{10^{-30}\ 3\times10^{8}} \sim 2\times10^{-12}\,m \qquad .02\,Å$$

impt for γ rays.

$\Phi, \vec{A}$ and gauge invariance

static $\vec{E} = -\nabla\phi$ $\quad\quad \vec{B} = \nabla\times\vec{A}$ (always, since $\nabla\cdot\vec{B} = 0$)

dynamic:

$$\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

$$= \nabla\times\left(\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = 0$$

$\therefore$ dynamic case $\quad -\nabla\phi = \vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

Gauge transform:

adding $\nabla\xi$ to $\vec{A}$ leaves $\vec{B}$ unaffected.

$$\vec{A}' = \vec{A} + \vec{\nabla}\xi \qquad \nabla\times\vec{A}' = \nabla\times\vec{A} + \underbrace{\cancel{\nabla\times(\vec{\nabla}\xi)}}_{0}$$

now $\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{A} + \vec{\nabla}\xi\right)$

$$= -\vec{\nabla}\left(\phi - \frac{1}{c}\frac{\partial\xi}{\partial t}\right) - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

$\therefore$ must also modify Φ:

$$\Phi' = \Phi - \frac{1}{c}\frac{\partial\xi}{\partial t}$$

$\nabla\times\vec{A} = \vec{B}$ doesn't fully specify $\vec{A}$

$\therefore$ add condition on $\nabla\cdot\vec{A}$

$= -\frac{1}{c}\frac{d\Phi}{dt}$ Lorentz gauge

$= 0$ coulomb gauge