Error Analysis in brief

“It is a well established fact of scientific investigation that the first time an experiment is performed the results often bear all too little resemblance to the “truth” being sought.
(…)
Whatever the reason, it is certainly true that for all physical experiments, errors and uncertainties exist that must be reduced by improved experimental techniques and repeated measurements, and that <u>these errors must always be estimated to establish the validity of our results</u>.”

P.R.Bevington and D.K.Robinson,
Data Reduction and Error Analysis for the physical sciences.

Precision VS Accuracy

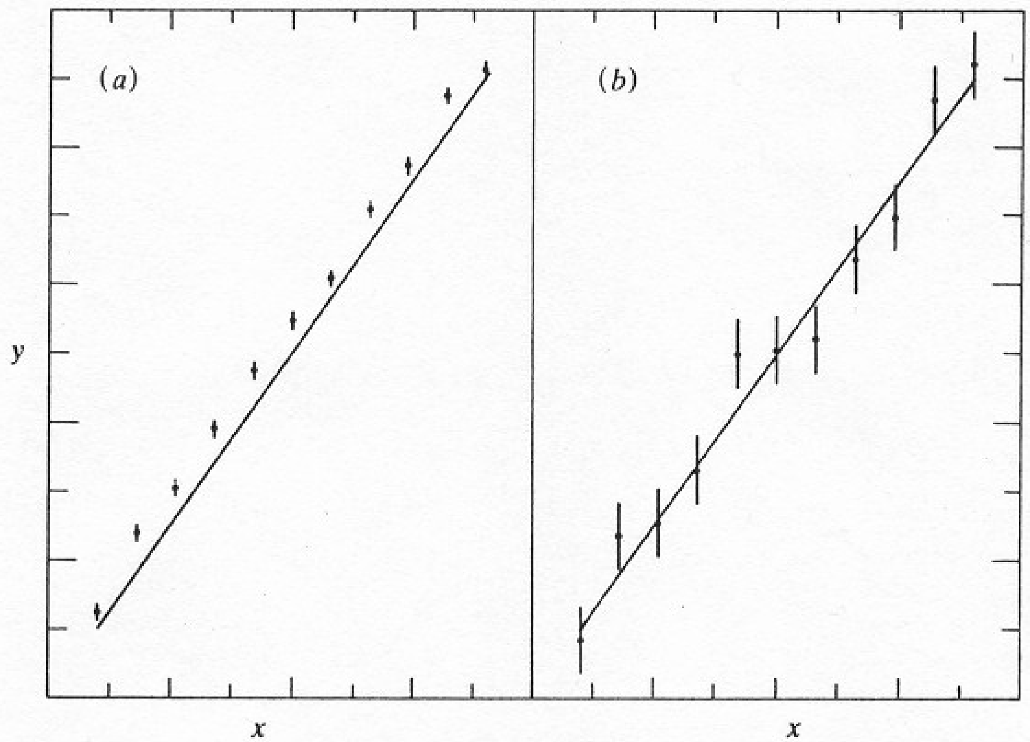


FIGURE 1.1
Illustration of the difference between precision and accuracy. (*a*) Precise but inaccurate data. (*b*) Accurate but imprecise data. True values are represented by the straight lines.

Errors affecting Accuracy and Precision

Accuracy:

depends on systematic errors.

These errors make our results different from the true values with reproducible discrepancy.

- difficult to detect and to estimate
- usually determined when another experiment with a different method measures the same property.

example: calibration error in equipment used.

Precision:

depends on random errors.

Random errors are the fluctuations in observations that yield results that differ from experiment to experiment and that require repeated experimentation to yield precise results.

→ Statistical distributions

Statistics of Measurements

If we make a measurement x, of a quantity X, we expect our observation to approximate the quantity, but we do not expect the experimental data point to be exactly equal to the quantity. Making more and more measurements, a pattern will emerge, grouped around the true value. This pattern can usually be described by a Gaussian Distribution around a mean value $\bar{X}$. If there are no systematical errors (or if we can correct for them), $\bar{X}$ should become the true value X for an infinite number of measurements.

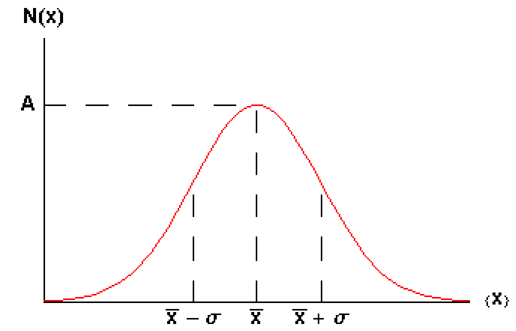
mean (value): $$\bar{X} = \frac{1}{N} \cdot \sum_{i=1}^{N} X_i$$

mean ($N \rightarrow \infty$): $$\mu = \lim_{N \rightarrow \infty} \left(\frac{1}{N} \cdot \sum_{i=1}^{N} X_i \right)$$

Dispersion of the observation → <u>Standard deviation σ (sigma)</u>

good measure for statistical error of a measurement.

Standard Deviation



Infinite number of measurements $N \rightarrow \infty$

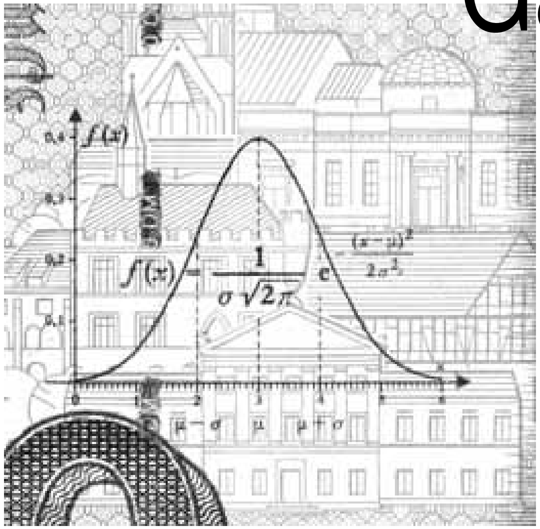
<u>Variance:</u> $$\sigma^2 = \lim_{N \rightarrow \infty} \left(\frac{1}{N} \cdot \sum_{i=1}^{N} (X_i - \mu)^2 \right)$$

In practice, one cannot make an infinite number of measurements:

$$s^2 = \frac{1}{N-1} \cdot \sum_{i=1}^{N} (X_i - \overline{X})^2$$

However, you will often found that s^2 is replaced by σ^2, especially when N>>1

Gaussian Distribution



The Gaussian distribution is widely used in statistical analysis of data, as it seems to describe the distribution of random observations for many experiments, as well as describing the distribution obtained when we try to estimate the parameters of most other probability distributions.

Probability function for Gaussian Distribution at X:

$$p(x_j) = \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp\left(-\frac{1}{2\sigma_j^2}\left(x_j - \hat{x}_j\right)^2\right),$$

where $\hat{x}_j$ is the predicted value of x_j

<u>How to proceed:</u>

- make measurements
- plot measurements
- fit with Gaussian Distribution
- X or μ give measured value
- σ gives statistical error

FWHM: Full Width Half Maximum

In practice, one need to determine σ “experimentally” from a set of data (measurements).

Easy way to do it: FWHM (Full Width at Half Maximum).

From FWHM, one can deduce the σ of the given distribution that fits the data best.

As, most of the time, one can fit the data with a Gaussian distribution

→ FWHM = Γ = 2.354 . σ

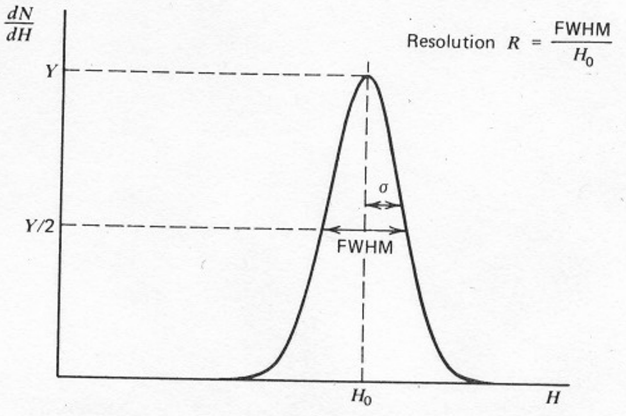


Figure 4-5 Definition of detector resolution. For peaks whose shape is Gaussian with standard deviation σ, the FWHM is given by 2.35σ.

A word on the binomial distribution

<u>Example:</u> Coin tosses “Heads” or “Tails”

Number of different possible combinations of n tosses giving x times the results “Heads”:

$$C(n,x) = \frac{1}{x!} \cdot \frac{n!}{(n-x)!}$$

The coins are indiscernible

How to get x “Heads” from n tosses.

Probability: p → “Heads” ; q → “Tails”

with: p+q=1, because there is 100% chance that a given coin toss will result in “Heads” or “Tails”

$$P(x,n,p) = C(n,x).p^{x}.q^{n-x} = C(n,x).p^{x}.(1-p)^{n-x}$$

One can show that:

<u>Mean:</u> $\mu = np$

<u>Standard Deviation:</u> $\sigma^2 = np(1-p)$

Application: p=q=0.5 (coin toss) → $\mu = n/2$; $\sigma^2 = n/4$

Poisson Distribution

In a given experiment, we need to evaluate our statistical uncertainties.

We have plenty of events: $n \gg 1$ and plenty of possibilities: $p_i \ll 1$, which also means that $n \gg \mu = np$ (see binomial distribution).

$$\sum_{i=1}^{\infty} p_i = 1$$

Problem: one cannot solve the problem exactly because all the p_i are usually not known. Fortunately, one can make approximations:

POISSON DISTRIBUTION: $$P_p(x,\mu) = \frac{\mu^x e^{-\mu}}{x!}$$

(do not depends on individual p_i's)

With (most importantly): $\sigma^2 = \mu \rightarrow \sigma = \sqrt{\mu}$

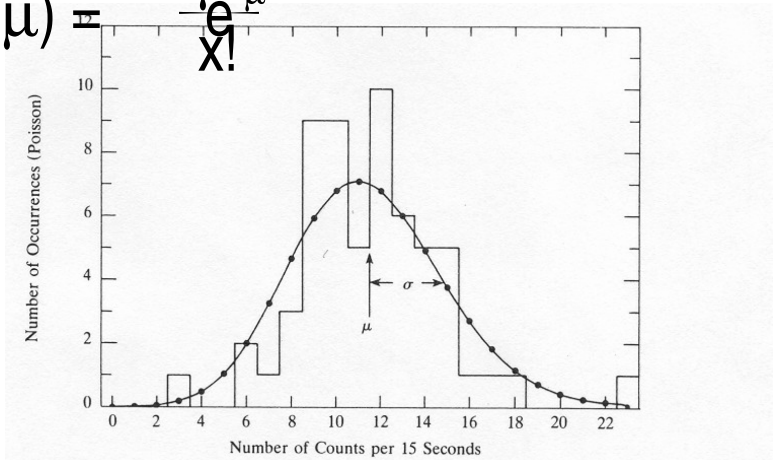


FIGURE 2.4
Histogram of counts in a cosmic ray detector. The Poisson distribution, shown as a continuous curve, is an estimate of the parent distribution based on the measured mean $\bar{x} = 11.48$. Only the circled calculated points are defined.

Application

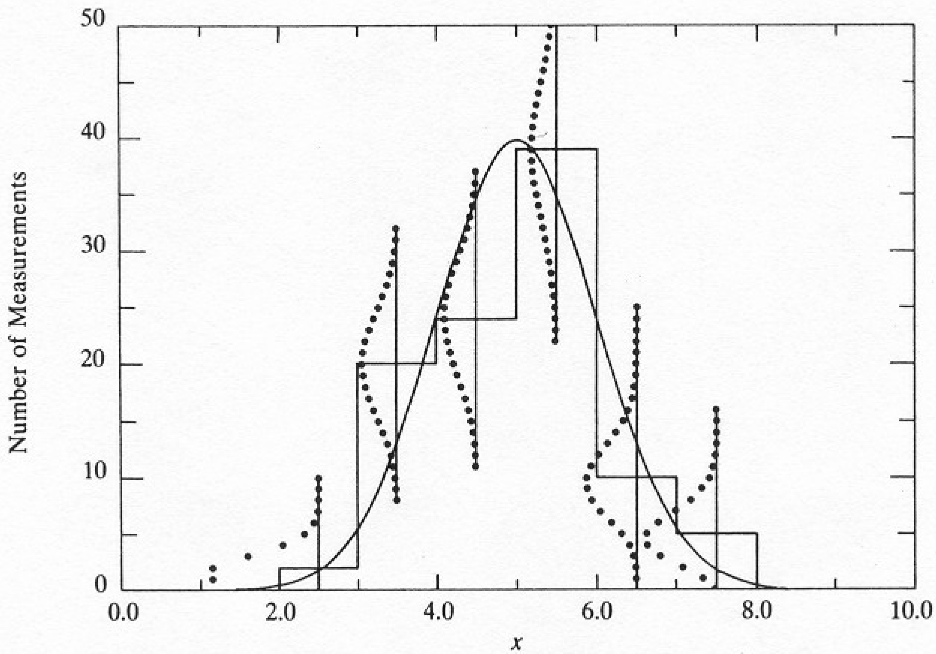


FIGURE 4.1
Histogram, drawn from a Gaussian parent distribution with mean $\mu = 5.0$ and standard deviation $\sigma = 1$, corresponding to 100 total measurements. The parent distribution $y(x_j) = NP(x_j)$ is illustrated by the large Gaussian curve. The smaller dotted curves represent the Poisson distribution of events in each bin, based on the sample data.

The Poisson distribution describes well the statistical error e.g. the error associated with the number of counts in each channel of a spectrum.

<u>Example:</u>
there is N_i counts in Channel i,
the associated error (at 1σ) is then:

$$\delta Ni = \sqrt{Ni}$$

→ Statistical Uncertainties
Statistical Fluctuations

Error Propagation (I)

Instrumental Uncertainties:

If the quantity x has been measured with a physical instrument, The uncertainty in the measurement generally comes from fluctuations in readings of the instrumental scale, either because the settings are not exactly reproducible due to the imperfections in the equipement, or because of human imprecision in observing settings (mostly a combination of both).

→ Measurements of: length
mass
voltage
current…

We get the uncertainty ($\sim\sigma$) by:

- estimate (know your instruments !)
- repeated measurements

Error Propagation (II)

In determining a certain dependent variable x that is a function of one or
More different measured variables, we need to combine our instrumental
And statistical uncertainties to find the uncertainty of the dependent variable x.

Assuming: $x = f(u,v,w)$

with $\sigma_u,\sigma_v,\sigma_w$ <u>independent</u> from each other.

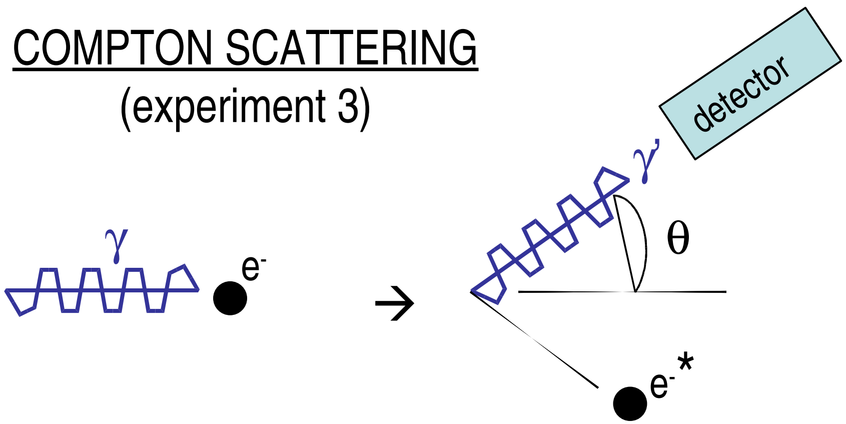
$$\Rightarrow \sigma_x^2 = \sigma_u^2 \cdot \left(\frac{\partial f}{\partial u}\right)^2 + \sigma_v^2 \cdot \left(\frac{\partial f}{\partial v}\right)^2 + \sigma_w^2 \cdot \left(\frac{\partial f}{\partial w}\right)^2$$

Special (but common) cases:

$x = f(u,v)$	$\rightarrow$	σ_x^2
$u + v$	$\rightarrow$	$\sigma_x^2 = \sigma_u^2 + \sigma_v^2$
$u \cdot v$	$\rightarrow$	$\sigma_x^2 = \left((\sigma_u/u)^2 + (\sigma_v/v)^2 \right) \cdot (u.v)^2$
u / v	$\rightarrow$	$\sigma_x^2 = \left((\sigma_u/u)^2 + (\sigma_v/v)^2 \right) \cdot (u/v)^2$

Summing Errors: an example

<u>COMPTON SCATTERING</u>
(experiment 3)



$$E_{\gamma'} = \frac{E_\gamma}{1 + (E_\gamma/m_0c^2)(1 - \cos\theta)}$$

You measure $\theta \pm d\theta$ and $E_\gamma \pm dE_\gamma$, what is the uncertainties on $E_{\gamma'}$ when you apply the above formula ?

Describe how $\sigma_{E\gamma'}$ is affected by $\sigma_{E\gamma}$ and σ_θ

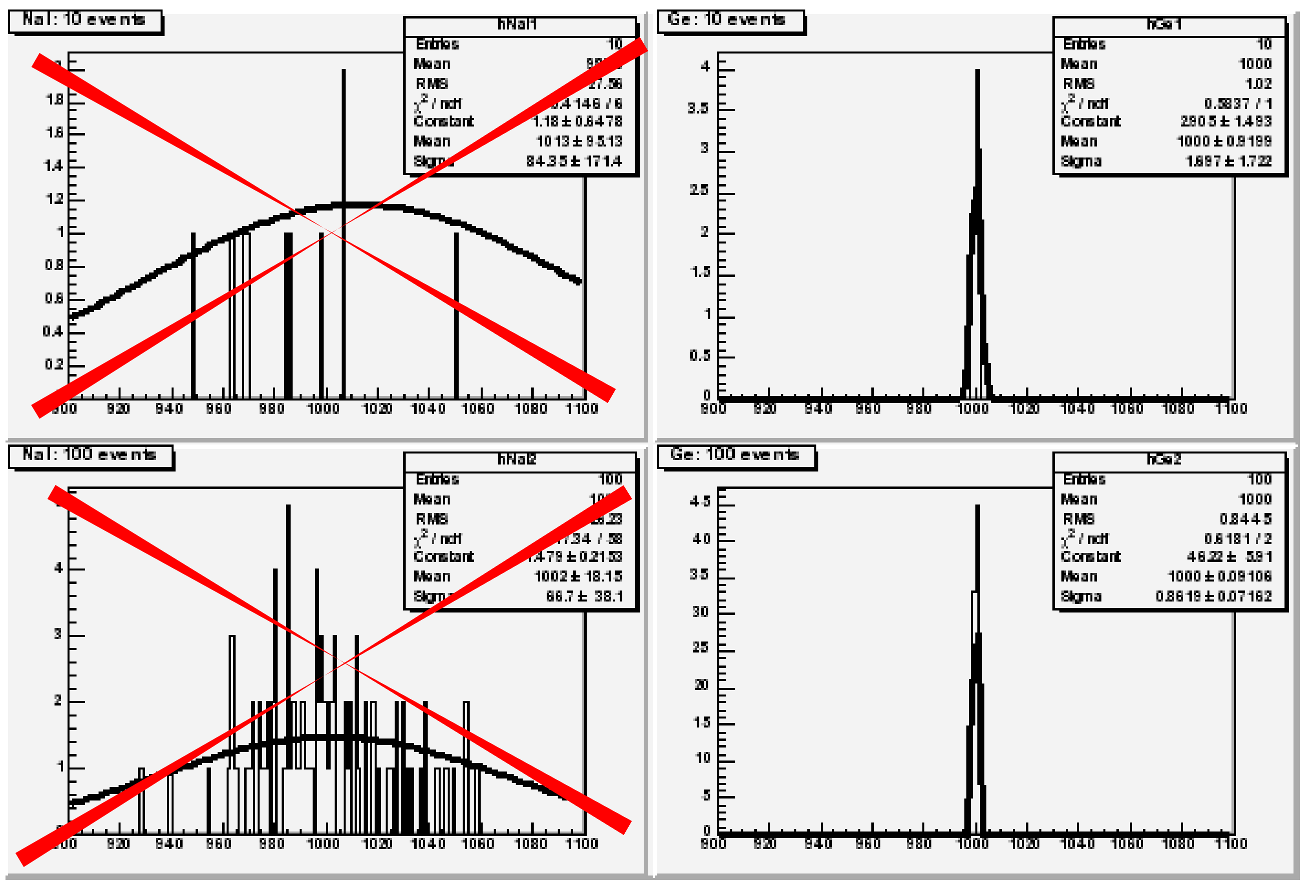
$$E_{\gamma'} = f(E_\gamma, \theta) \quad \Rightarrow \sigma_{E\gamma'}^2 = \sigma_{E\gamma}^2 \cdot \left(\frac{\partial f}{\partial E_\gamma}\right)^2 + \sigma_\theta^2 \cdot \left(\frac{\partial f}{\partial \theta}\right)^2$$

Independent errors

Statistics & Resolution: NaI VS Germanium detector (I)

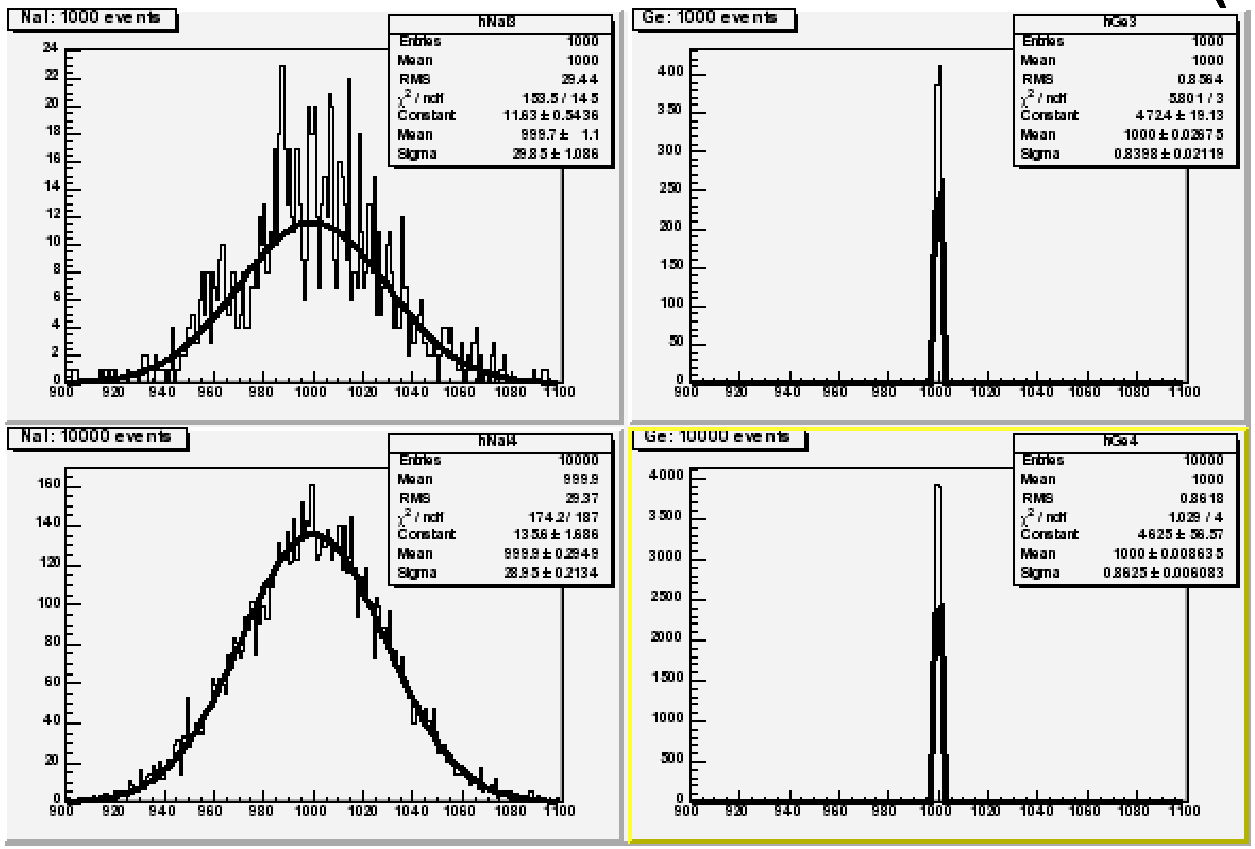
As seen before, the resolution of a Germanium detector is better than a NaI detector.
Typically:

NaI: $\frac{\Delta E}{E}$ ~7% (70 keV @ 1 MeV) Ge: $\frac{\Delta E}{E}$ ~0.5% (2 keV @ 1 MeV)



Statistic too low =
fit cannot be trusted !
(Background !)

Statistics & Resolution: NaI VS Germanium detector (II)

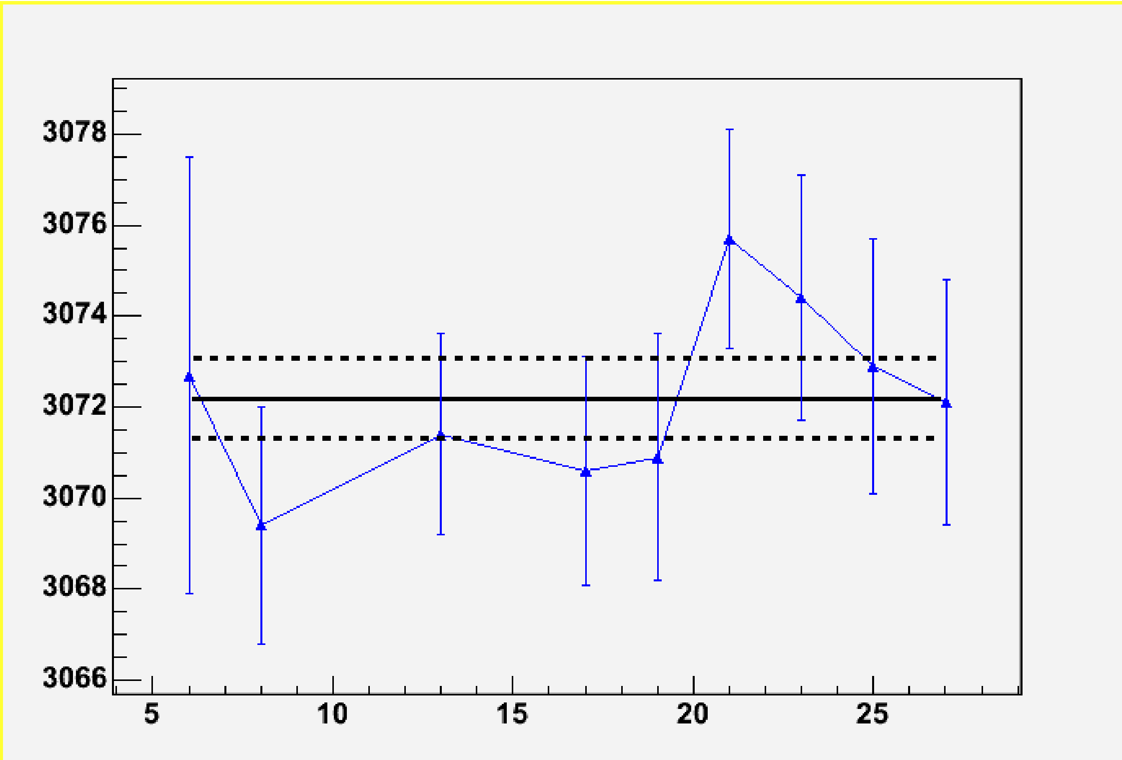


The better the resolution, the less number of events one requires to make a measurement at a given precision…

→ make sure you collect <u>an appropriate amount of events</u> before doing the analysis of your data

Summing "identical" measurements

(See Experiment 2)



Measurements (data set):

6 3072.7(48) = 3072.7± 4.8
8 3069.4(26)
13 3071.4(22)
17 3070.6(25)
19 3070.9(27)
21 3075.7(24)
23 3074.4(27)
25 3072.9(28)
11 3072.1(27)

x_i σ_i

→ Mean value:

3072.2(8) = 3072.2 ± 0.8

$$\bar{X} = \frac{\sum_{i=1}^{N} (x_i / \sigma_i^2)}{\sum_{i=1}^{N} (1/\sigma_i^2)}$$

Weighted Average

$$\bar{\sigma}^2 = \frac{1}{\sum_{i=1}^{N} (1/\sigma_i^2)}$$

Smaller (better) final precision

References…

Nuclear Physics:

Introductory Nuclear Physics – Kenneth S. Krane

Radiation Detection and Measurement:

Radiation Detection and Measurement – Glenn F. Knoll

Error Analysis:

Data Reduction and Error Analysis for the Physical Sciences
– Philip R. Bevington & D. Keith Robinson