PHGN 462 Homework 3

1) There's a general method for describing the polarization of electromagnetic waves using what are called Jones vectors. We'll restrict ourselves to describing the electric field in a wave, since once you know E and the direction in which the wave is traveling, you can easily find the orientation and amplitude of B (at least, I hope you can).

Let's say we have an E-field propagating along the z-axis with the form

$$\vec{E} = E_x e^{i(kz-wt-\varphi_x)}\hat{\imath} + E_y e^{i(kz-wt\quad y)}\hat{\jmath}$$

Such a form is totally general, and allows for the possibility that the $\hat{\imath}$ and $\hat{\jmath}$ components of the field are of different amplitudes and also of different phases with respect to one another. We can factor out the part common to both components and write $\vec{E}$ in vector form as follows:

$$\vec{E} = e^{i(kz-\quad)}\begin{pmatrix} E_x e^{i\varphi_x} \\ E_y e^{i\varphi_y} \end{pmatrix}$$

The term in parentheses has all the information that's unique to a particular field, showing both the component amplitudes and phases. This term is the Jones vector.

The Jones vectors for horizontally and vertically polarized light with unit amplitude are $\vec{E}_H = \binom{1}{0}$ and $\vec{E}_V = \binom{0}{1}$, respectively. This pair defines a simple basis that can be used to express the electric field for any wave.

a) Consider the E-field whose real part is $\vec{E} = 7\cos(kz - wt)\,\hat{\imath} - 5\sin(kz - wt)\,\hat{\jmath}$ (with implied units attached to the 7 and 5). Figure out how to decompose this field in terms of $\vec{E}_H$ and $\vec{E}_V$. By that I mean find the A and B such that $\vec{E} = A\vec{E}_H + B\vec{E}_V$. You may find it helpful to start by expressing $\vec{E}$ in terms of a Jones vector. And don't forget that you can use imaginary coefficients if you need to.

b) An alternative basis for describing polarization is referred to as circular polarization. The basis vectors (in Jones notation) are $\vec{E}_L = \frac{1}{\sqrt{2}}\binom{1}{i}$ and $\vec{E}_R = \frac{1}{\sqrt{2}}\binom{1}{-i}$, describing left-circular and right-circular polarization respectively.

Explain, using an appropriate combination of words, equations, and diagrams, why this basis is referred to as *circular* polarization. It's no great challenge to find the answer on the interwebs, so make sure your explanation is strong and is uniquely your own. Also mention why those $\frac{1}{\sqrt{2}}$ factors are there.

c) Express the E-field from part (a) in terms of $\vec{E}_L$ and $\vec{E}_R$

Two things to take away from this problem: 1) How to express polarization in general and 2) That circular polarization, while it sometimes sounds like an odd thing, is just another basis to work in, one that happens to come up a lot in optics labs.

2) We've seen plane waves and spherical waves (or will see them very soon), and they're very nice and all, but it'd be really fantastic if there was a more confined solution to the wave equation. A set of fields that traveled along some axis without being infinite in extent. A *beam*, if you will. Might such a thing exist?

a) A beam would have certain properties. For example, it'd propagate primarily along a single axis, in much the same way that a plane wave does. That means some component of the associated E-field would have a form such as:

$$E(x,y,z,t) = u(x,y,z)e^{i(kz-\omega t)}$$

where u is some as-yet-unknown spatial distribution and we're defining $k = \frac{\omega}{c}$.

For a beam, u would probably have certain properties, too. It's likely that it'd vary slowly along the direction of propagation, barely changing at all on a length scale similar to λ. And its variance in the axial direction would be small compared to its variance in transverse directions. Together those features are encoded in the so-called paraxial approximations:

$$\left|\frac{\partial^2 u}{\partial z^2}\right| \ll \left|\frac{\partial^2 u}{\partial x^2}\right|, \left|\frac{\partial^2 u}{\partial y^2}\right| \qquad \text{and} \qquad \left|\frac{\partial^2 u}{\partial z^2}\right| \ll \left|k\frac{\partial u}{\partial z}\right|$$

Using all of the above, show that the field envelope u for a beamlike solution to the wave equation will satisfy:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2ik\frac{\partial u}{\partial z} = 0$$

b) The above can be solved in a rather brute force fashion via separation of variables, leading to an answer written in terms of Hermite polynomials. It's a *very* long process, though, so I'll just give you the solution:

$$E = E_{n,m}H_n\left(\frac{\sqrt{2}x}{w(z)}\right)H_m\left(\frac{\sqrt{2}y}{w(z)}\right)\frac{w_0}{w(z)}\cdot e^{\frac{-(x^2+y^2)}{w(z)^2}}\cdot e^{-i\frac{kr^2}{2R(z)}}\cdot e^{i(1+n+m)\tan^{-1}\frac{z}{z_r}}\cdot e^{i(kz-\omega t)}$$

where *w(z)* is the spot size defined by, $w(z) = w_0\sqrt{1+\left(\frac{z}{z_r}\right)^2}$, *R(z)* is the radius of curvature , $R(z) = z\left(1+\frac{z_r^2}{z^2}\right)$, and the constants w_0 and z_r are called the beam waist and Rayleigh length, which are tied together via $z_r = \frac{\pi w_0^2}{\lambda}$. H_n is the n^{th} Hermite polynomial.

Each combination of n and m leads to a distinct allowed mode. $(n,m) = (0,0)$ is what's known as the TEM_{00} mode, and is the mode in which we frequently try to operate lasers.

There's a ton of physics in the above equation, often involving tradeoffs between one beam parameter or another. For example: Take a look at the mode equation for TEM_{00} modes and tell me how focusing the beam affects collimation of the beam. That is, if I focus the beam down to a smaller waist, does that result in a more or less well-collimated beam? Explain how you know.

c) Let's get a look at these modes by plotting the E-fields in Mathematica (or whatever platform you prefer). Plot *xz* and *xy* cross sections of a few different modes and comment on what you see. For the *xz* modes, show a snapshot at a particular time as opposed to a time average, so we can see the oscillations along *z*. For the *xy* cross sections, do as you please.

Note that you might have some dynamic range issues… the field values involved span enough orders of magnitude that Mathematica's default color mapping will probably wash out some detail. If it's too washed out to be useful, fix it. I used a logarithmic color mapping. The code for that was relatively easy to find on the interwebs.