MATH348-Advanced Engineering Mathematics		**Homework: Fourier Series - Part II**
Complex Representation, Resonant Forcing, Bessel's Inequality, Parseval's Identity		
Text: 11.3-11.4	Lecture Notes : 9-10	Lecture Slides: N/A

Quote of Homework Five

Carlos Castaneda: I had not been using my eyes. That was true, yet I was very sure he had said to feel the difference. I brought that point up, but he argued that one can feel with the eyes, when the eyes are not looking right into things.

Carlos Castaneda - The Teachings of Don Juan: A Yaqui Way of Knowledge (1968)

1. Fourier Series : Nonstandard Period

Let $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$ be such that $f(x+4) = f(x)$.

1.1. **Graphing.** Sketch f on $(-4, 4)$.

1.2. **Symmetry.** Is the function even, odd or neither?

1.3. **Integrations.** Determine the Fourier coefficients a_0, a_n, b_n of f.

1.4. **Truncation.** Using `http://www.tutor-homework.com/grapher.html`, or any other graphing tool, graph the first five terms of your Fourier Series Representation of f.

2. Fourier Series : Periodic Extension

Let $f(x) = \begin{cases} \dfrac{2k}{L}x, & 0 < x \leq \frac{L}{2} \\ \\ \frac{2k}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$.

2.1. **Graphing - I.** Sketch a graph f on $[-2L, 2L]$.

2.2. **Graphing - II.** Sketch a graph f^*, the even periodic extension of f, on $[-2L, 2L]$.

2.3. **Fourier Series.** Calculate the Fourier cosine series for the half-range expansion of f.

3. Complex Fourier Series

3.1. **Orthogonality Results.** Show that $\left\langle e^{inx}, e^{-imx} \right\rangle = 2\pi\delta_{nm}$ where $n, m \in \mathbb{Z}$, where $\langle f, g \rangle = \displaystyle\int_{-\pi}^{\pi} f(x)g(x)dx$.

3.2. **Fourier Coefficients.** Using the previous orthogonality relation find the Fourier coefficients, c_n, for the complex Fourier series, $f(x) = \displaystyle\sum_{n=-\infty}^{\infty} c_n e^{inx}$.

3.3. **Complex Fourier Series Representation.** Find the complex Fourier coefficients for $f(x) = x^2,\ -\pi < x < \pi,\ f(x + 2\pi) = f(x)$.

3.4. **Conversion to Real Fourier Series.** Using the complex Fourier series representation of f recover its associated real Fourier series.

4. Periodic Forcing of Simple Harmonic Oscillators

Consider the ODE, which is commonly used to model forced simple harmonic oscillation,

$$y'' + 9y = f(t), \tag{1}$$

$$f(t) = |t|, \quad -\pi \leq t < \pi, \quad f(t+2\pi) = f(t). \tag{2}$$

Since the forcing function (2) is a periodic function we can study (1) by expressing $f(t)$ as a Fourier series. [1] [2]

4.1. **Fourier Series Representation.** Express $f(t)$ as a real Fourier series.

4.2. **Method of Undetermined Coefficients.** The solution to the homogeneous problem associated with (1) is $y_h(t) = c_1\cos(3t) + c_2\sin(3t),\ \ c_1, c_2 \in \mathbb{R}$. Knowing this, if you were to use the method of undetermined coefficients[3] then what would your choice for the particular solution, $y_p(t)$? DO NOT SOLVE FOR THE UNKNOWN CONSTANTS

4.3. **Resonant Modes.** What is the particular solution associated with the third Fourier mode of the forcing function?[4]

4.4. **Structural Changes.** What is the long term behavior of the solution to (1) subject to (2)? What if the ODE had the form $y'' + 4y = f(t)$?

5. Error Analysis and Applications

We have that for a *reasonable* $2\pi-$periodic function there exist coefficients a_0, a_n, b_n such that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx). \tag{3}$$

This is, of course, the Fourier series representation of the function f but, as we know, computational devices are not well-suited to infinite sums. Thus, we would like to know how f is approximated by

$$f(x) \approx f_N(x) = a_0 + \sum_{n=N}^{\infty} a_n\cos(nx) + b_n\sin(nx), \tag{4}$$

were this $N^{th}-$partial sum is called a trigonometric polynomial. Since f_N approximates f on an interval, we define our error as

$$E = \int_{-\pi}^{\pi}(f - f_N)^2 dx, \tag{5}$$

which is called *the squared error* of f_N.[5] It can be shown that this squared error can be written as

$$E = \int_{-\pi}^{\pi} f^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right]. \tag{6}$$

It is plausible that as $\lim_{N\to\infty} f_N = f$ and $E \to 0$. Thus, from (6) we have

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 dx, \tag{7}$$

which is called Parseval's identity.[6]

5.1. **Application of Mean Square Error.** Let $f(x) = x^2$ for $x \in (-\pi, \pi)$ such that $f(x + 2\pi) = f(x)$. Determine the value of N so that $E < 0.001$.

5.2. **Application of Parseval's identity.** Using the previous function and Parseval's identity show that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.

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[1] The advantage of expressing $f(t)$ as a Fourier series is its validity for any time t. An alternative approach have been to construct a solution over each interval $n\pi < t < (n+1)\pi$ and then piece together the final solution assuming that the solution and its first derivative is continuous at each $t = n\pi$.

[2] It is worth noting that this concepts are used by structural engineers, a sub-disciple of civil engineering, to study the effects of periodic forcing on buildings and bridges. In fact, this problem originate from a textbook on structural engineering.

[3] This is also known as the method of the 'lucky guess' in your differential equations text.

[4] Each term in a Fourier series is called a mode. The first mode is sometimes called the fundamental mode. The rest of the modes, called *harmonics* in acoustics, are just referenced by number. The third Fourier mode would be the third term of Fourier summation

[5] We choose to square the integrand so that there can be no possible cancellation of positive errors/areas with negative errors/areas.

[6] These are the main equations associated with the error analysis of Fourier series. A student interested in the derivations should consult Kreyszig's section 11.4, 9th edition.