MATH348-Advanced Engineering Mathematics		Homework: PDEs - Part III
Wave Equations: Traveling and Standing Waves, Nodal Lines, and Nonlinear Equations		
Text: 12.2, 12.8	Lecture Notes : N/A	Lecture Slides: N/A

Quote of Homework Six
Our vibrations were getting nasty. But why? Was there no communication in this car? Had we deteriorated to the level of dumb beasts?
Duke : Fear and Loathing in Las Vegas (1998)

1. D'Alembert Solution to the Wave Equation in $\mathbb{R}^{1+1}$

Show that by direct substitution the function $u(x,t)$ given by,

(1) $$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(y)dy,$$

is a solution to the one-dimensional wave equation where u_0 and v_0 are the ideally elastic objects initial displacement and velocity, respectively. [1]

2. Wave Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the one-dimensional wave equation,

(2) $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2},$$

(3) $$x \in (0,L), \quad t \in (0,\infty), \quad c^2 = \frac{T}{\rho}.$$

Equations (2)-(3) model the time-evolution of the displacement from rest, $u = u(x,t)$, of an elastic medium in one-dimension. The object, of length L, is assumed to have a homogeneous lateral tension T, and linear density ρ. That is, $T, \rho \in \mathbb{R}^+$. Assume, as well, the boundary conditions[2],

(4) $$u_x(0,t) = 0, u_x(L,t) = 0,$$

and initial conditions,

(5) $$u(x,0) = f(x),$$

(6) $$u_t(x,0) = g(x).$$

2.1. **Separation of Variables : General Solution.** Assume that the solution to (2)-(3) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (2)-(3), which satisfies (4)-(6). [3] [4]

2.2. **Qualitative Dynamics.** Describe how the the general solution to (2)-(3) changes as the tension, T, is increased while all other parameters are held constant. Also, describe how the solution changes when the linear density, ρ, is increased while all other parameters are held constant.

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[1] This is called the d'Alembert solution to the wave equation. To do this you may want to recall the fundamental theorem of calculus, $\frac{d}{dx}\int_0^x f(t)dt = f(x)$ and properties of integrals, $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$.

[2] These boundary conditions imply that the object must have zero slope at each endpoint.

[3] It is important to notice that the solution to the spatial portion of the problem is the same as the heat problem above.

[4] Remember that in this case we have a nontrivial spatial solution for zero eigenvalue. From this you should find the associated temporal function should find that $G_0(t) = C_1 + C_2 t$.

2.3. **Fourier Series : Solution to the IVP.** Define,

$$f(x) = \begin{cases} \dfrac{2k}{L}x, & 0 < x \le \frac{L}{2}, \\ \dfrac{2k}{L}(L-x), & \frac{L}{2} < x < L. \end{cases} \tag{7}$$

Let $L = 1$ and $k = 1$ and find the particular solution, which satisfies the initial displacement, $f(x)$, given by (7) and has zero initial velocity for all points on the object.

3. Inhomogeneous Wave Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the non-homogeneous one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x,t) \quad , \tag{8}$$

$$x \in (0, L), \qquad t \in (0, \infty), \qquad c^2 = \frac{T}{\rho}. \tag{9}$$

with boundary conditions and initial conditions,

$$u(0,t) = u(L,t) = 0, \tag{10}$$

$$u(x,0) = u_t(x,0) = 0. \tag{11}$$

Letting $F(x,t) = A\sin(\omega t)$ gives the following Fourier Series Representation of the forcing function F,

$$F(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{L}\right), \tag{12}$$

where

$$f_n(t) = \frac{2A}{n\pi}(1 - (-1)^n)\sin(\omega t). \tag{13}$$

3.1. **Educated Fourier Series Guessing.** Based on the boundary conditions we assume a Fourier sine series solution. However, the time-dependence is unclear. So, assume that,

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) G_n(t), \tag{14}$$

where $G_n(t)$ represents the unknown dynamics of the n-th Fourier mode. Using this assumption and (12)-(13), show by direct substitution that (8) yields the ODE,

$$\ddot{G}_n + \left(\frac{cn\pi}{L}\right)^2 G_n = \frac{2A}{n\pi}(1 - (-1)^n)\sin(\omega t). \tag{15}$$

3.2. **Solving for the Dynamics.** The solution to (15) is given by,

$$G_n(t) = G_n^h(t) + G_n^p(t), \tag{16}$$

where $G_n^h(t) = B_n \cos\left(\frac{cn\pi}{L}t\right) + B_n^* \sin\left(\frac{cn\pi}{L}t\right)$ is the homogeneous solution and $G_n^p(t)$ is the particular solution to (15).

3.2.1. *Particular Solution - I.* If $\omega \neq cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS

3.2.2. *Particular Solution - II.* If $\omega = cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS

3.2.3. *Physical Conclusions.* For the Particular Solution - II, what is $\lim_{t\to\infty} u(x,t)$ and what does this limit imply physically?

4. Vibrations of a Rectangular Membrane: Wave Equation on a Bounded Domain of $\mathbb{R}^{2+1}$

Suppose that you are given an infinitesimally thin, ideally elastic membrane of area $A = L_x L_y$, which is allowed to move in the z-axis direction but is permanently fixed along its perimeter. Use the solution to the corresponding PDE to describe the first four fundamental vibrational modes and the structure of their nodal lines.

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Given

(I) $u_{tt} = c^2 u_{xx}$, $x \in (0,L)$, $t \in (0,\infty)$, $c^2 = T/\rho$ | Local String Acceleration is proportional to local concavity

(II) $u_x(0,t)=0$, $u_x(0,t)=0$ | End points may move but must stay flat.

(III) $u(x,0) = f(x)$ | Initial Shape

$u_t(x,0) = g(x)$ | Initial Velocity

<u>Step I</u>: Notice that nothing has changed from Eqn (I) from class.

Thus, $u(x,t) = X(x)T(t) \Rightarrow$

$$\Rightarrow u_{tt} = X\ddot{T} = X''Tc^2 = u_{xx}c^2$$

$X \neq 0$, $T \neq 0 \Rightarrow$

$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X} = \underbrace{-\lambda \in \mathbb{R}}_{\text{Separation Constant}} \quad (*)$$

<u>Key Argument</u>: If the LHS is a fn of t and the RHS a fn of x <u>and</u> they must be Equal for all t,x then they must <u>not</u> be fn of t <u>or</u> x.

Key Outcome: (*) gives 2 sets of ODE

$$X'' + \lambda X = 0$$

$$\ddot{T} + c^2 \lambda T = 0$$

parameterized by λ.

Step II: Now things have changed b/c (II) are not the same as class.

$$u_x(0,t) = \left.\frac{\partial u}{\partial x}\right|_{x=0} = X'(0)T(t) = 0$$

Dynamics $\Rightarrow X'(0) = 0$

$\Rightarrow u_x(L,t) = 0 \Rightarrow X'(L) = 0$

thus our BVP is

$$X'' + \lambda X = 0, \ \lambda \in \mathbb{R}$$

such that $X'(0) = 0, \ X'(L) = 0$

We have the 3 sets of general sol$\underline{n}$ that use a total of six fn

$$\lambda > 0: \ X_1(x) = C_1 \sin(\sqrt{\lambda}\, x) + C_2 \cos(\sqrt{\lambda}\, x)$$

$$\lambda < 0: \ X_2(x) = C_3 \sinh(\sqrt{|\lambda|}\, x) + C_4 \cosh(\sqrt{|\lambda|}\, x)$$

$$\lambda = 0: \ X_3(x) = C_5 x + C_6$$

Now $X'(0) = 0 \Rightarrow C_1 = C_5 = 0$

$X'(L) = 0 \Rightarrow C_4 = 0$

thus,

$$\lambda > 0: \ X_1'(L) = -C_2 \sqrt{\lambda} \sin(\sqrt{\lambda}\, L) = 0$$

$$\Rightarrow \sqrt{\lambda_n} = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots$$

$$\Rightarrow X_n(x) = C_n \cos(\sqrt{\lambda_n}\, x), \quad C_n \in \mathbb{R}$$

Note

$\lambda = 0; \ X_3(x) = C_6$ always satisfies the B.C.

or $X_0(x) = C_0$ [Use zeros for convention]

We now have a set of spatial sol<u>n</u> and a set of angular freq. ~~and~~ So, we go ~~k~~ back to the time problem, which remains unchanged from the class notes. Thus,

$$T_n(t) = A_n\cos(c\sqrt{\lambda_n}\,t) + B_n\sin(c\sqrt{\lambda_n}\,t)$$

Well, there is actually 1 change and that is the introduction of $\lambda=0$ as a freq. In this case

$$T'' + c^2 \underbrace{\cancel{\lambda}}_{0} T = T'' = 0$$

$$\Rightarrow \quad T_0(t) = A_0 + \cancel{A_1}\, B_0 t$$

<u>Key Outcome</u>: General Sol<u>n</u>

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) =$$

$$= X_0(x)T_0(t) + \sum_{n=1}^{\infty} \cancel{\cos}\, X_n(x)\,T_n(t) =$$

$$= A_0 + B_0 t + \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n}\,x)\left[A_n\cos(c\sqrt{\lambda_n}\,t) + B_n\sin(c\sqrt{\lambda_n}\,t)\right]$$

Key Point: This is just the same soln as before except the B.C. changed the ~~spati~~ type of spatial waves!

Note:

- $X_0(x) = \text{constant}$ can be thought of a wave with $\emptyset$ freq. or ∞-wavelength.

  Also, this is cosine of $\sqrt{\lambda_0} = \frac{0 \cdot \pi}{L}$.

Step III: This will be the same as class but now we note the relation

$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx \overset{\substack{\text{Even simplification}\\ \uparrow}}{=} 2\int_{0}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx$$

$$= L \underbrace{\delta_{nm}}_{\text{Kronecker Delta fn}} = \begin{cases} L, & m = n \\ 0, & m \neq n \end{cases}$$

$$\Rightarrow \int_{0}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = \frac{L}{2}\delta_{nm}, \quad \text{for integer } m, n$$

Thus, $u(x,0) = f(x)$ implies

$$\int_0^L \underbrace{u(x,0)}_{f(x)} \cos\left(\frac{m\pi}{L}x\right)dx = \qquad \cos\left(\frac{m\pi}{L}x\right)dx$$

$$= \int_0^L \left[A_0 + \cancelto{0}{B_0 t} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right)\left[A_n \cancelto{1}{\cos(0)} + B_n \cancelto{0}{\sin(0)}\right]\right] \cdot$$

$$= A_0 \int_0^L \cos\left(\frac{m\pi}{L}x\right)dx + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx}_{\frac{L}{2}\delta_{nm}} =$$

$$= \frac{A_0 L}{m\pi} \cancelto{0-0=0}{\sin\left(\frac{m\pi}{L}x\right)\Big|_0^L} + \sum_{n=1}^{\infty} A_n \frac{L}{2}\delta_{nm} = A_m \frac{L}{2} \cancelto{1}{\delta_{mm}}$$

$$\Rightarrow \boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx,}$$

the m was an arbitrary integer so we just call it n for ease of use.

Also,

$$\int_0^L u(x,0)\,dx = A_0\int_0^L dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi}{L}x\right)dx$$

$$= A_0\cdot L + \underbrace{\sum_{n=1}^{\infty} \frac{A_n L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\Big|_0^L}_{0-0=0} \Rightarrow \boxed{A_0 = \frac{1}{L}\int_0^L f(x)\,dx}$$

Now, the initial velocity says,

$$\int_0^L u_t(x,0)\cos\left(\frac{m\pi}{L}x\right)dx =$$

$$= \int_0^L \left[B_0 + \sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{L}x\right)\left[c\sqrt{\lambda_n}A_n \overset{0}{\sin(0)} + B_n c\sqrt{\lambda_n}\underset{1}{\cos(0)}\right]\right]\cos\left(\frac{m\pi}{L}x\right)dx$$

$$= B_0\underbrace{\int_0^L \cos\left(\frac{m\pi}{L}x\right)dx}_{0-0=0} + \sum_{n=1}^{\infty} B_n c\sqrt{\lambda_n}\underbrace{\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx}_{\frac{L}{2}\delta_{nm}}$$

$$= B_m c\sqrt{\lambda_m}\,\frac{L}{2}\,\delta_{mm} \Rightarrow \boxed{B_n = \frac{2}{c\sqrt{\lambda_n}}\int_0^L g(x)\cos\left(\frac{n\pi}{L}x\right)dx}$$

Lastly, a similar argument gives,

$$\boxed{B_0 = \frac{1}{L}\int_0^L u_t(x,0)\,dx.}$$

Eqn (!!) with all Green boxes Represents the soln to the initial boundary value Problem (I)-(III).

Notes:

- Suppose $f(x)=0$, for all x and $g(x)>0$ for all x. Then
  $A_0 = A_n = 0$ for all n,
  and
  $B_0 > 0$
  thus the displacement grows in time. That is, the ~~#~~ spring moves "up" forever. AKA yours ~~the~~ there's it.

Notes Cont:

- T, ρ control time-freq. as before.

- There is a new Equilibrium (Rest) state where the string is initially ~~flat~~. only flat

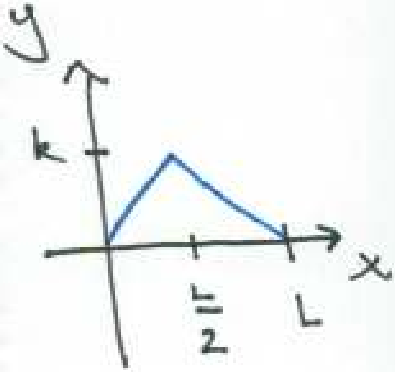
  $U(x,0) = \alpha \in \mathbb{R} \Rightarrow A_n = 0$ for all n.

  $U_t(x,0) = 0 \Rightarrow B_0 = B_n = 0$

  $\Rightarrow U(x,t) = \alpha$ [This was always zero for our fixed end cond.]

If $g(x) = 0$ for all x and

$$f(x) = \begin{cases} \frac{2k}{L}x, & x \in (0, \frac{L}{2}) \\ \frac{2k}{L}(L-x), & x \in (\frac{L}{2}, L) \end{cases}$$



then $B_0 = B_n = 0$ and

$$A_0 = \frac{1}{L}\int_0^L u(x,0)\,dx = \frac{1}{L}\left[\frac{1}{2}\cdot L\cdot k\right] =$$

$$= \frac{k}{2}$$

$$A_n = \frac{2}{L}\int_0^L u(x,0)\cdot\cos\left(\frac{n\pi}{L}x\right)dx =$$

$$= \frac{2}{L}\left[\int_0^{L/2} \frac{2k}{L}\overset{u_1}{x}\,\underbrace{\cos\left(\frac{n\pi}{L}x\right)dx}_{dv} + \int_{L/2}^{L}\frac{2k}{L}\overbrace{\left(L-x\right)}^{u_2}\underbrace{\cos\left(\frac{n\pi}{L}x\right)dx}_{dv}\right]$$

$$= \frac{4k}{L^2}\left[\cancel{\frac{L}{2n\pi}\sin\left(\frac{n\pi}{2}\right)} + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{L^2}{n^2\pi^2} - \cancel{\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right)} + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{L^2}{n^2\pi^2}\right]$$

$$= \frac{8k}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] = \begin{cases} \dfrac{-8k}{n^2\pi^2}, & n = 1,3,5,\cdots \\ 0, & n = 4,8,12,\cdots \\ \dfrac{-16k}{n^2\pi^2} & n = 2,6,10,\cdots \end{cases}$$

u_1	u_2	dv
x +	$L-x$ +	$\cos\left(\frac{n\pi}{L}x\right.$
1 −	-1 −	$\frac{L}{n\pi}\sin\left(\right.$
0	0	$\frac{-L^2}{n^2\pi^2}\cos\left(\right.$