


11-9-07

Note Title

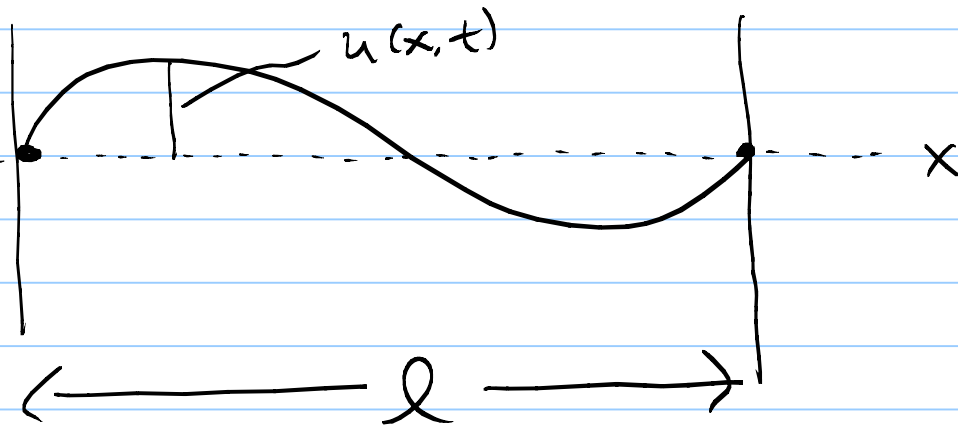
11/7/2006

important Fourier Series example
Solving B.V.P. with sep. of variables

Eg. 
$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

$\underbrace{\hspace{10em}}$
Laplacian in
1D

This models the vibrations
of a string clamped
at the ends



Boundary conditions

$$u(0,t) = u(l,t) = 0 \quad \text{clamped end}$$

initial conditions

$$u(x, 0) = u_0(x)$$

$$\frac{du}{dt}(x, 0) = v_0(x)$$

Guess

Suppose we can

find a solution of the form:

$$u(x, t) = \underline{X}(x) \underline{T}(t) \text{ not obviously true!}$$

Give it a try and see if it works

$$\frac{\partial u}{\partial x} = \underline{X}' \underline{T}$$

prime = space deriv,

dot = time deriv,

$$\frac{\partial u}{\partial t} = \underline{X} \dot{\underline{T}}$$

So

$$\text{flag} \quad \frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$\Rightarrow \bar{X}(x) \ddot{T}(t) = c^2 \bar{X}''(x) T(t)$$

Always divide by $\bar{X}T$ on
Both sides

$$\Rightarrow \frac{\bar{X} \ddot{T}}{\bar{X}T} = c^2 \frac{\bar{X}'' T}{\bar{X}T}$$

$$\Rightarrow \frac{\ddot{T}}{T} = c^2 \frac{\bar{X}''}{\bar{X}}$$

depends only
on t

depends
only on x

How?

only if

$$\frac{\ddot{T}}{T} = c^2 \frac{\ddot{X}}{X} = \text{constant}$$

This constant must have dimensions $\left[\frac{1}{\text{time}}\right]^2$

so it's a frequency squared.

$$\frac{\ddot{T}}{T} = c^2 \frac{\ddot{X}}{X} = -\omega^2$$

↑
minus sign will be expl. later



$$\ddot{T} + \omega^2 T = 0$$

$$\underline{\underline{X'' + \frac{\omega^2}{c^2} X = 0}}$$

$$\Rightarrow T(t) = A \cos(\omega t) + B \sin(\omega t)$$

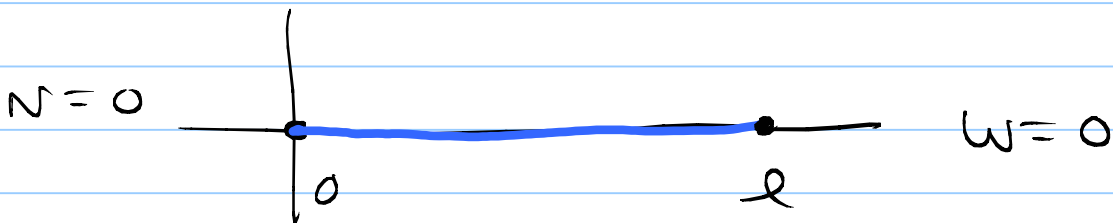
$$\Rightarrow \underline{X(x)} = C \cos\left(\frac{\omega}{c} x\right) + D \sin\left(\frac{\omega}{c} x\right)$$

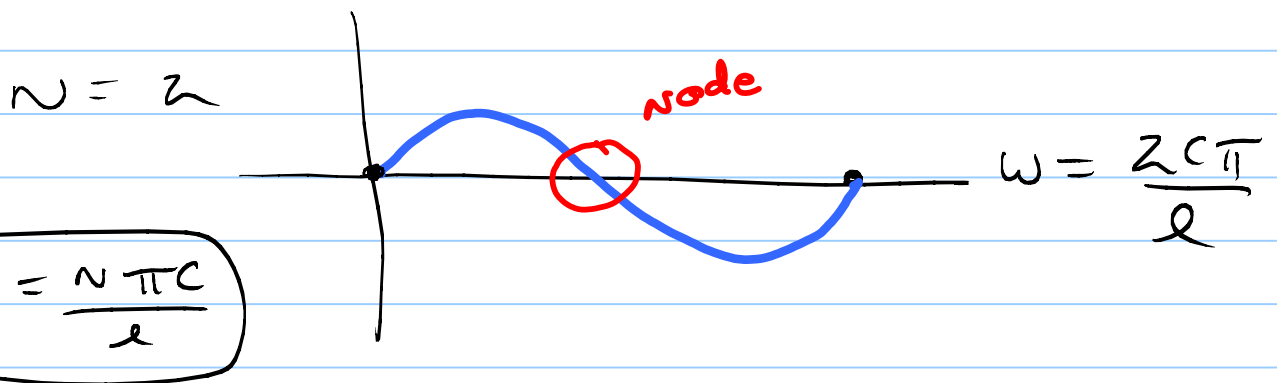
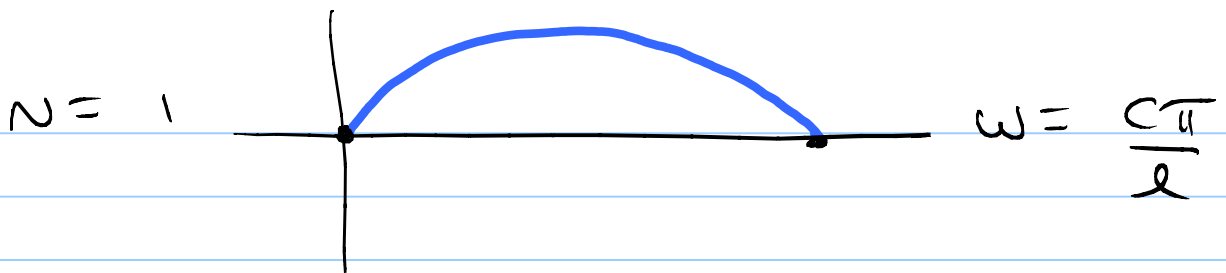
Apply B.C.

$$\underline{X(0)} = 0 \Rightarrow C = 0$$

$$\underline{X(l)} = 0 \Rightarrow \frac{\omega l}{c} = n\pi$$

$$\text{so } \underline{X(x)} = D \sin\left(\frac{n\pi}{l} x\right)$$





$$\omega_N = \frac{N\pi c}{l}$$

notice that $\omega \uparrow$ as $c \uparrow$

and $\omega \downarrow$ as $l \uparrow$

longer string

stiffer string

The spatial solutions

$$X_N(x) = D \sin(N\pi x/l)$$

are called modes

Now the time equation
 $T(t) = A \cos(\omega t) + B \sin(\omega t)$

The BC gives us a
quantization condition
on frequency.

$$X(l) = 0 \Rightarrow \frac{\omega l}{c} = n\pi$$

$$\Rightarrow \omega = \frac{n\pi c}{l}$$

Lets call this ω_n to

emphasize the discreteness

$$\omega_n = \frac{n\pi c}{l}$$

So

$$T(t) = A \cos(\omega_n t) + B \sin(\omega_n t)$$

$$u(x,t) = \underline{X(x)} T(t)$$

$$\begin{aligned} \text{recall } \underline{X(x)} &= D \sin\left(\frac{n\pi x}{l}\right) \\ &= D \sin\left(\frac{\omega_n}{c} x\right) \end{aligned}$$

So

$$u(x,t) = \textcircled{AD} \cos(\omega_n t) \sin\left(\frac{\omega_n}{c} x\right) + \textcircled{BD} \sin(\omega_n t) \sin\left(\frac{\omega_n}{c} x\right)$$

A

B

$$u(x,t) = A \cos(\omega_n t) \sin\left(\frac{\omega_n}{c} x\right) + B \sin(\omega_n t) \sin\left(\frac{\omega_n}{c} x\right)$$

Does this satisfy the initial conditions?

$$u(x,0) = A \sin\left(\frac{\omega_n}{c} x\right)$$

not in general

Let

$$u_1(x,t) = A_1 \cos(\omega_1 t) \sin\left(\frac{\omega_1}{c} x\right) + B_1 \sin(\omega_1 t) \sin\left(\frac{\omega_1}{c} x\right)$$

$$u_2(x,t) = A_2 \cos(\omega_2 t) \sin\left(\frac{\omega_2}{c} x\right) + \dots$$

Since the PDE

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

is Linear

$$\frac{\partial^2}{\partial t^2} [u_1 + u_2] = c^2 \frac{\partial^2}{\partial x^2} [u_1 + u_2]$$

$u_1 + u_2$ is also a solution

But now we have 4 const,
to help us satisfy the

Initial conditions

Keep going

Let

$$u(x,t) = \sum_{N=0}^{\infty} A_N \sin\left(\frac{\omega_N}{c} x\right) \cos(\omega_N t) + B_N \sin\left(\frac{\omega_N}{c} x\right) \sin(\omega_N t)$$

Now try the I. Cond.

$$u(x,0) = \sum_{N=0}^{\infty} A_N \sin\left(\frac{\omega_N}{c} x\right) \equiv u_0(x)$$

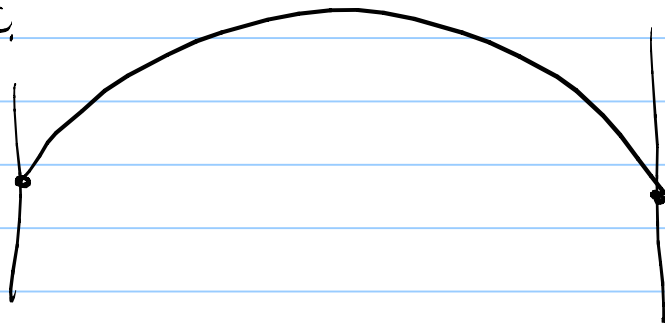
i.e. we can satisfy one of the I.C. if we can represent $u_0(x)$ as a Fourier Series

How about $\frac{\partial u}{\partial t} \Big|_{t=0} \equiv v_0$

$$V_0(x) = \sum_{n=0}^{\infty} \omega_n B_n \sin\left(\frac{\omega_n}{c} x\right)$$

SUPPOSE we pluck a string

Ex.



Snapshot at $t = 0$

$$u(x, 0) = A_1 \sin\left(\omega_1/c x\right) \quad \begin{array}{l} n=1 \\ \text{mode} \end{array}$$

$$= A_1 \sin(\pi x/l)$$

$$= \sum_{n=0}^{\infty} A_n \sin\left(\frac{\omega_n}{c} x\right)$$

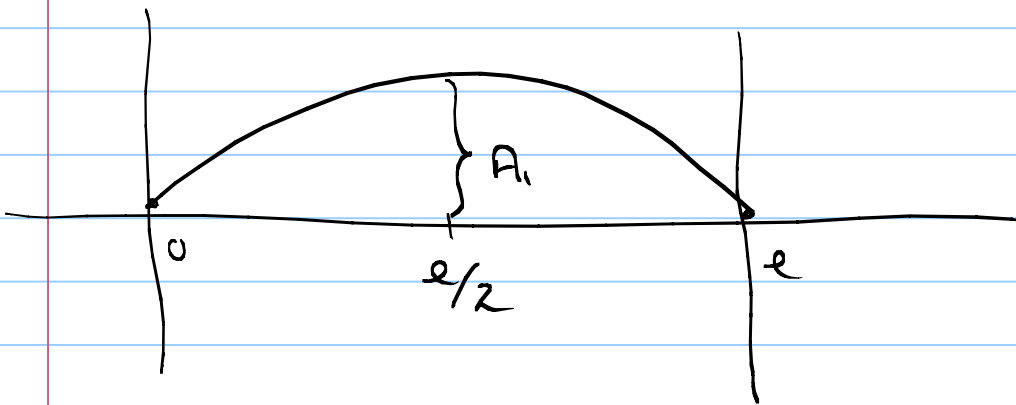
$$\Rightarrow A_n = \begin{cases} A_1 & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases}$$

$$v(x, 0) = 0 \Rightarrow B_n = 0$$

$$\Rightarrow u(x,t) = A_1 \sin\left(\frac{\omega_1}{c}x\right) \cos(\omega_1 t)$$

$$u(x,t) = A_1 \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi c t}{l}\right)$$

complete solution



$$u(l/2, 0) = A_1 \sin\left(\frac{\pi}{2}\right) = A_1$$

So A_1 is specified by how far you pull the string back.