MATH348: SPRING 2012 - HOMEWORK 4

FOURIER TRANSFORMS, CONVOLUTION AND GREEN'S FUNCTIONS

The answer is within the problem.

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Abstract. A Fourier series is this thing that takes in *reasonable* periodic data and represents the data through the use of sinusoids in linear combination. That is, if f is a reasonable $2L-$periodic function then the linear combination

$$f(x) = \sum_{n=-\infty}^{\infty} c(\omega_n) e^{i\omega_n x} \tag{1}$$

whose coefficients are given by

$$c(\omega_n) = \frac{1}{2L}\left\langle f, e^{i\omega_n x}\right\rangle = \frac{1}{2L}\int_a^b f(x)e^{-i\omega_n x}dx \tag{2}$$

where $b > a$, $b - a = 2L$ and $\omega_n = n\pi/L$ is the complex Fourier series representation of the periodic function f. The following is how to think about the previous formulae:

1. The sinusoids in the linear combination (1) are called *Fourier modes.* Specifically, the $k^{th}-$mode is given by the pair
$$\begin{aligned} c_{-k}e^{i\omega_{-k}x} + c_k e^{i\omega_k x} &= c_k^* e^{-i\omega_k x} + c_k e^{i\omega_k x} \\ &= \left(\frac{a_k + ib_k}{2}\right)e^{-i\omega_k x} + \left(\frac{a_k - ib_k}{2}\right)e^{i\omega_k x} \\ &= a_k\left(\frac{e^{-i\omega_k x} + e^{i\omega_k x}}{2}\right) + b_k i\left(\frac{e^{-i\omega_k x} - e^{i\omega_k x}}{2}\right) \\ &= a_k\cos(\omega_k x) + b_k\sin(\omega_k x). \end{aligned}$$
2. Since the Fourier modes are also waves, the superposition of waves gives rise to constructive and destructive interference of waves. Thus, the data $f(x)$ can be thought of as the interference created by the sum in Eq. (1). Well, so long as the amplitudes are picked just right.
3. Equation (2) projects the data f onto each of the basis waves $e^{i\omega_n x}$ so that the interference pattern created by the sum in Eq. (1) gives the original data. Physically we can think of each wave/mode and it's amplitude/coefficient in the following ways:
   - FM: In electromagnetism $f(x)$ is the radiation and a mode can be thought of as a color in the E+M spectrum. In acoustics $f(t)$ is the signal/sound and a mode can be thought of a specific note.
   - FC: In E+M the Fourier coefficient represents the intensity of a specific color and in acoustics the coefficient represents the loudness of a specific note.

   From this, and a comparison to the simple harmonic oscillator, we can speak about the energy in the data via the relation
$$E \propto \sum_{n=-\infty}^{\infty} |c(\omega_n)|^2 \tag{3}$$
   and now the Fourier series is this thing that takes in an energetically reasonable function and splits it into primitive periodic functions and amplitudes of oscillation.

The basic problem we now have is that to use Fourier series we have to restrict the frequencies to a discrete set of values. We take this to mean that we can only speak about the notes given by the keys of a piano and not those notes we know to exist between the piano's keys. To remedy this problem we note that $\Delta\omega = \pi/L$ will shrink as $L \to \infty$. Under this limit the equations (1)-(3) become

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega, \quad \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx, \quad E \propto \int_{-\infty}^{\infty}|\hat{f}(\omega)|^2 d\omega, \tag{4}$$

which is the well-celebrated Fourier transform pair, which inherits all of the previous thinking about Fourier series, but can be used whether or not the data f is periodic. The following problems illustrate some general properties and common transforms associated with Fourier transform:

- P1. Here we take Fourier transforms of various types of functions. In class we will give interpretation of each.
- P2. Recall that when we started talking about Fourier series, symmetry did most of the heavy lifting. We then considered complex Fourier series as the tool of choice if symmetry is nonexistent or not important for the problem at hand. This notation also makes the transition to complex Fourier transform fairly straightforward. With this problem we take the time to again consider symmetry as it applies to Fourier transform.
- P3. The sine/cosine Fourier transform pair can be thought of, in keeping with the discussion of the abstract, analogous to sine/cosine half-range expansions.
- P4. The transform of a product is not the product of transforms, it is, instead, a convolution integral.
- P5. Convolutions are natural when solving DEs with transform methods.

1. Fourier Transforms

Calculate the following transforms. Include any domain restrictions.

1.1. **Dirac-Delta.** $\mathfrak{F}\left\{f\right\}$ where $f(x) = \delta(x - x_0),\ \ x_0 \in \mathbb{R}$.[1]

1.2. **Even Combination of Dirac-Deltas.** $\mathfrak{F}^{-1}\left\{\hat{f}\right\}$ where $\hat{f}(\omega) = \frac{1}{2}\left(\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right),\ \ \omega_0 \in \mathbb{R}$.

1.3. **Odd Combination of Dirac-Deltas.** $\mathfrak{F}^{-1}\left\{\hat{f}\right\}$ where $\hat{f}(\omega) = \frac{1}{2}\left(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right),\ \ \omega_0 \in \mathbb{R}$.

1.4. **Decaying Exponential Function.** $\mathfrak{F}\left\{f\right\}$ where $f(x) = e^{-k_0|x|},\ \ k_0 \in \mathbb{R}^+$.

1.5. **Horizontal Shifts.** Find $\mathfrak{F}\left\{f(x + c)\right\}, c \in \mathbb{R}$.

1.6. **Horizontal Stretch.** Find $\mathfrak{F}\left\{f(ax)\right\}, a \in \mathbb{R}^+$.

1.7. **Differentiation.** Show that if both f and f' have Fourier transforms then $\mathfrak{F}\left\{f'\right\} = i\omega\mathfrak{F}\left\{f\right\}$.[2]

1.8. **Exponential Function.** Show that if $f(x) = e^{-ax}U_0(x)$ then $\hat{f}(\omega) = \dfrac{1}{\sqrt{2\pi}(a + i\omega)}$ where $a \in \mathbb{R}^+$.[3]

2. Fourier Transforms of Symmetric Functions

Let,

$$f_c(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}(\omega)\cos(\omega x)d\omega, \quad \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\omega x)dx \tag{6}$$

$$f_s(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}(\omega)\sin(\omega x)d\omega, \quad \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\omega x)dx, \tag{7}$$

be the definitions for the Fourier cosine and Fourier sine transform pairs, receptively.

2.1. **Symmetry.** Show that $f_c(x)$ and $\hat{f}_c(\omega)$ are even functions and that $f_s(x)$ and $\hat{f}_s(\omega)$ are odd functions.[4]

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[1] Here the δ is the so-called Dirac, or continuous, delta function. This isn't a function in the true sense of the term but instead what is called a generalized function. The details are unimportant and in this case we care only that this Dirac-delta *function* has the property $\int_{-\infty}^{\infty}\delta(x - x_0)f(x)dx = f(x_0)$. For more information on this matter consider `http://en.wikipedia.org/wiki/Dirac_delta_function`. To drive home that this *function* is strange, let me spoil the read. The sampling function $f(x) = \text{sinc}(ax)$ can be used as a definition for the Delta *function* as $a \to 0$. So can a bell-curve probability distribution. Yikes!

[2] Remember writing down the Fourier transform required absolute integrability of the transformed function and that absolute integrability implied that the function decay at infinity. This should help with the 'uv' term from integration by parts.

[3] Here we use the Heavyside function

$$U_0(x) = \begin{cases} 1, & x \in [0, \infty), \\ 0, & x \in (-\infty, 0), \end{cases} \tag{5}$$

which restricts the our exponential function to the left side of the plane.

[4] Thus, if an input function has an even or odd symmetry then the transformed function shares the same symmetry.

2.2. **Derivation from Fourier Transform.** Recall the complex Fourier transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega, \quad \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx. \tag{8}$$

Show that if we assume that $f(x)$ is an even function then (8) defines the transform pair given by (6). Also, show that if $f(x)$ is an odd function then (8) defines the transform pair given by (7). [5]

2.3. **Even and Odd Finite Pulses.** Given,

$$f(x) = \begin{cases} A, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}, \qquad A, a \in \mathbb{R}^+. \tag{9}$$

Plot the even and odd extensions of f.

2.4. **Symmetric Transforms.** Find the Fourier cosine and sine transforms of f.

2.5. **Integral Trick.** Using the Fourier cosine transform show that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(\pi\omega)}{\pi\omega} d\omega = 1$.

3. Sine and Cosine Transforms

Calculate the following Fourier sine/cosine transformations. Include any domain restrictions.

3.1. **Forward Cosine Transform.** $\mathfrak{F}_c(e^{-ax}),\ a \in \mathbb{R}^+$

3.2. **Inverse Cosine Transform.** $\mathfrak{F}_c^{-1}\left(\dfrac{1}{1+\omega^2}\right)$

3.3. **Forward Sine Transform.** $\mathfrak{F}_s(e^{-ax}),\ a \in \mathbb{R}^+$

3.4. **Inverse Sine Transform.** $\mathfrak{F}_s^{-1}\left(\sqrt{\dfrac{2}{\pi}}\dfrac{\omega}{a^2+\omega^2}\right),\ a \in \mathbb{R}^+$

4. Convolution Integrals

The convolution h of two functions f and g is defined as[6],

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(p) g(x-p) dp = \int_{-\infty}^{\infty} f(x-p) g(p) dp. \tag{10}$$

4.1. **Fourier Transforms of Convolutions.** Show that $\mathfrak{F}\{f * g\} = \sqrt{2\pi}\mathfrak{F}\{f\}\mathfrak{F}\{g\}$. [7]

4.2. **Simple Convolution.** Find the convolution $h(x) = (f * g)(x)$ where $f(x) = \delta(x - x_0)$ and $g(x) = e^{-x}$.

5. Simple Green's Functions

Given the ODE,

$$y' + y = f(t), \quad 0 < t < \infty. \tag{11}$$

5.1. **Delta Forcing.** Calculate the frequency response, or what is sometimes called the steady-state transfer function, associated with (11). [8]

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[5] Thus, if an input function has symmetry then the Fourier transform is real-valued and reduced to a sine or cosine transform.

[6] Here wee keep the same notation as Kreysig pg. 523

[7] The point here is that while the Fourier transform of a linear combination is the linear combination of Fourier transforms, the Fourier transform of a product is a convolution integral. Well, at least that's something.

[8] This function is a representation of how the system responds to the most primitive force, $\delta(t)$, in the Fourier domain.

5.2. **Inversion.** Calculate the Green's function associated with (11). [9]

5.3. **Solutions as Convolution Integrals.** Using convolution find the steady-state solution to the (11).[10]

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[9]The Green's function is just the inverse of the frequency response function and is a representation of how the system would like to respond to the primitive, $\delta(t)$, force in the original domain.

[10]The key point of these three steps is that, if you can determine how a linear differential equation responds to simple forcing then the general solution can be represented in terms of a convolution integral containing the Green's function. This gives us a general method for solving inhomogeneous differential equations. However, finding the Green's function for a DE is generally nontrivial.