Lecture 21: Momentum in electromagnetic fields

We've established that the "lengths" of four-vectors are invariant under Lorentz transforms. So, for example,

$$p^\mu p_\mu = -p_0 p_0 + p_1 p_1 + p_2 p_2 + p_3 p_3 = -E^2/c^2 + \vec{p}\cdot\vec{p}$$

Must be the same in every frame. We came up with p^μ as a description of the 4-momentum of a massive particle, but what if we rather cavalierly think of it as applying to something massless, like an EM wave?

An EM wave has an energy density, so some chunk of it has a well-defined $-E^2/c^2$. In another frame it has some other $-E^2/c^2$, meaning that if we can describe fields with p^μ or something like it, then the invariance of $p^\mu p_\mu$ suggests fields carry non-zero momentum.

Let's see if we can be a bit more rigorous. Fields + momentum (for matter) are related via the Minkowski version of $\vec{F} = d\vec{p}/dt = q(\vec{E} + \vec{v}\times\vec{B})$:

$$\frac{dp^\mu}{d\tau} = q\eta_\nu F^{\mu\nu}$$

Now say instead of a point charge we have some charge density in motion (which is to say, a current density).

Classically, $J = \rho v$, so $J d^3x = \rho d^3x\, v = dq\, v$

Generalizing to Minkowski space, we substitute $q\eta_\nu = J_\nu d^3x$

$\Rightarrow$ $$\frac{dp^\mu}{d\tau} = J_\nu F^{\mu\nu} d^3x$$ Divide across by the volume element and the left hand side becomes a statement about momentum density for the charge density on the right

$\Rightarrow$ $$\frac{d\wp^\mu}{d\tau} = J_\nu F^{\mu\nu}$$

Now, every conserved quantity has a continuity equation.

If charge leaves a volume, there must be divergent current:

$$\nabla\cdot\vec{J} = -\partial\rho/\partial t$$

If energy leaves a volume, there must be divergent energy flux:

$$\nabla \cdot \vec{Q} = -\partial u / \partial t$$

And if momentum leaves a volume, there must be some kind of divergent momentum flux:

$$\nabla \cdot \vec{T} = -\partial \wp / \partial t$$

So in Minkowski formulation:

$$\frac{d\wp^{\mu}}{d\tau} = J_{\nu} F^{\mu\nu} = \text{some divergence}$$

The divergence of what? $\frac{\partial}{\partial x^{\nu}}$ is the divergence operator in Minkowski space. It needs to act on something and return a 4-vector to be consistent with the above.

$\frac{\partial}{\partial x^{\nu}}$ acting on a 4-vector returns a scalar:

$$\frac{\partial}{\partial x^{\nu}} \text{☺}^{\nu} = \frac{\partial \text{☺}^0}{\partial x^0} + \frac{\partial \text{☺}^1}{\partial x^1} + \dots = \text{number}$$

So to be consistent with the above, we need the divergence to act on a tensor (rank 2), since the divergence operator contracts our object by one rank:

$$\frac{d}{d\tau} \wp^{\mu} = J_{\nu} F^{\mu\nu} = -\frac{\partial}{\partial x^{\nu}} T^{\mu\nu}$$

scalar operator

4 vector

generates a 4-vector

4 vector

rank 2 tensor

generates a 4-vector through matrix mult.

4-vector operator

Contracts a rank 2 tensor to make 4-vector

So $J_{\nu} F^{\mu\nu}$ must equal $-\frac{\partial T^{\mu\nu}}{\partial x^{\nu}}$ for some tensor $T^{\mu\nu}$

in order for 4-momentum to have its continuity equation. This kind of suggests $T^{\mu\nu}$ is built out of J_{ν} and $F^{\mu\nu}$, so let's break it down:

For $\mu = 0$,

$$J_\nu F^{0\nu} = -\frac{\partial T^{0\nu}}{\partial x^\nu}$$

$$\Rightarrow \quad \cancel{J_0 F^{00}}^{\,0} + \underbrace{J_1 F^{01} + J_2 F^{02} + J_3 F^{03}}_{\vec{J}\cdot\vec{E}/c} = -\frac{1}{c}\frac{\partial}{\partial t}T^{00} - \frac{\partial}{\partial x_1}T^{01} - \frac{\partial}{\partial x_2}T^{02} - \frac{\partial}{\partial x_3}T^{03}$$

This'll give us a hook into what the components of T are, since back in Chpt. 11 we found that $\vec{J}\cdot\vec{E}$ shows up in Poynting's thm, which is itself a continuity equation:

$$\nabla\cdot\vec{S} = -\frac{\partial u_{field}}{\partial t} - \vec{E}\cdot\vec{J}$$

$$\Rightarrow \quad \frac{\vec{J}\cdot\vec{E}}{c} = -\frac{1}{c}\frac{\partial u_{field}}{\partial t} - \frac{\partial}{\partial x_1}\frac{S_1}{c} - \frac{\partial}{\partial x_2}\frac{S_2}{c} - \frac{\partial}{\partial x_3}S_3/c$$

Comparing, it's clear that $\quad T^{00} = u_{field} \quad$ and $\quad T^{0i} = \frac{S_i}{c}$

and, altogether $\quad J_\nu F^{\mu\nu} = -\frac{\partial T^{\mu\nu}}{\partial x^\nu}$

has Poynting's theorem as its $\mu = 0$ component.

If the 0^{th} component of this thing expresses the local conservation of energy, what do you want to wager the ith component expresses the local conservation of momentum?

Let's look at $\mu = i$:

$$J_\nu F^{i\nu} = -\frac{\partial}{\partial x^\nu}T^{i\nu}$$

$$\Rightarrow \quad J_0 F^{i0} + J_1 F^{i1} + \ldots = -\frac{1}{c}\frac{\partial}{\partial t}T^{i0} - \frac{\partial}{\partial x}T^{i1} - \ldots$$

$$\Rightarrow \quad c\rho F^{i0} + J_1 F^{i1} + \ldots = -\frac{1}{c}\frac{\partial}{\partial t}T^{i0} - \frac{\partial}{\partial x}T^{i1} \ldots$$

The $J_1 F^{i1}$ etc terms pick out the B portions of $F^{\mu\nu}$ and we end up with:

$$\underbrace{\rho E^i}_{cF^{i0}} + \underbrace{\varepsilon_{ijk} J^j B^k}_{(\vec{J}\times\vec{B})_i} = -\frac{1}{c}\frac{\partial}{\partial t}T^{i0} - \frac{\partial T^{ij}}{\partial x^j} \qquad \text{(A)}$$

So what is that LHS?

$$c\rho E_i + (\vec{J}\times\vec{B})_i \qquad \text{and} \quad \vec{J} = \rho\vec{v}$$

$$\Rightarrow \underbrace{c\rho E_i + \rho(\vec{v}\times\vec{B})_i}$$

This is the Lorentz force law, but for densities ρ instead of point charges!

And force expresses the rate at which we add momentum to a system, so we could write a momentum continuity equation as

$$\frac{\partial P_{mech}}{\partial t} + \frac{\partial P_{field}}{\partial t} = -\vec{\nabla}\cdot(\text{momentum flux})$$

Or

$$\frac{\partial P_{mech}}{\partial t} = -\frac{\partial P_{field}}{\partial t} - \vec{\nabla}\cdot(\text{momentum flux})$$

Comparing to (A), we identify

$$-\frac{1}{c}\frac{\partial}{\partial t}T^{i0} = -\frac{\partial P_{field_i}}{\partial t} \Rightarrow T^{i0} = i\text{th contribution to field momentum density}$$

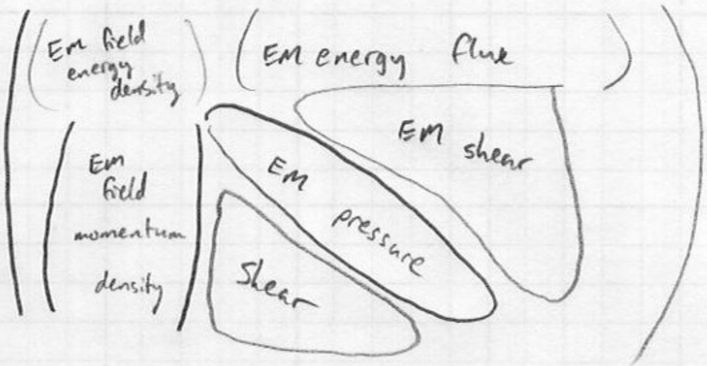
$$-\frac{\partial T_{ij}}{\partial x_j} = i\text{th contribution to divergence of momentum flux}$$

$$\Rightarrow T_{ij} = \text{flux of } j\text{th component of field momentum in } i\text{th direction.}$$

Now, transfer of i-momentum in the i-direction is a force. Per area, it's a pressure, so T_{ii} tells you about electromagnetic pressures.

Transfer of i-momentum in the j-direction $(i\neq j)$ is a shear force, so the off-diagonal T_{ij}'s are about shear.

Qualitatively, we have:

$T =$ 

Some of the $T^{\mu\nu}$ we know in terms of E, B.
Some we don't yet. All of them can be inferred from

$$J_\nu F^{\mu\nu} = -\frac{\partial T^{\mu\nu}}{\partial x^\nu}$$

Or, even more fundamentally, you can get $T^{\mu\nu}$ from a Lagrangian formulation of E+M.

$T^{\mu\nu}$ is called the EM stress-energy tensor. There are lots of stress-energy tensors; this is just the one for EM fields. In detail one can work out that

$$T^{\mu\nu} = \begin{pmatrix} \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2 & S_x/c & S_y/c & S_z/c \\ S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ S_z/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix}$$

where $\sigma_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)\delta_{ij}$

The attentive reader will note that this is symmetric and in particular, the $T^{0i} = T^{i0}$, despite having different physical labels in the above! That's because for an EM wave energy flux and momentum density are <u>the same thing</u> up to a factor of c, reproducing the quantum result $E = pc$, at least in spirit.