Note Title 9/29/2006

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$$\min_{\vec{x}} \|A\vec{x}-\vec{y}\|^2$$

$$= \min_{\vec{x}} \underbrace{(A\vec{x}-\vec{y},\ A\vec{x}-\vec{y})}$$

$$(A\vec{x}, A\vec{x}) - (\vec{y}, A\vec{x}) - (AX, \vec{y}) + (\vec{y}, \vec{y})$$

claim $\nabla_{\vec{x}} (\underbrace{\vec{y}, A\vec{x}}) = A^T \vec{y}$

$$\sum_i y_i [A\vec{x}]_i$$

$$= \sum_i y_i \sum_j A_{ij} x_j$$

$$= \sum_{i,j} A_{ij} y_i x_j$$

$$= \sum_{i,j} x_j \ A_{ij} y_i$$

$$= \sum_{i,j} x_j A^T_{ji} y_i$$

$$= \sum_{j} x_j \underbrace{\sum_{i} A^T_{ji} y_i}_{(A^T \vec{y})_j}$$

$$= (\vec{x}, A^T y)$$

So $\left[\nabla_{\vec{x}} (\vec{y}, A\vec{x}) \right]_k$

$$\frac{\partial}{\partial x_k} \left[\sum_{j} x_j (A^T \vec{y})_j \right]$$

zero except for $j = k$

$$= (A^T \vec{y})_k$$

$$\nabla_{\vec{x}} (\vec{y}, A\vec{x}) = A^T y$$

$$\nabla_{\vec{x}}$$

$$(A\vec{x}, A\vec{x}) - \underbrace{(\vec{y}, Ax) - (Ax, y)}_{\text{Took care of these just now}} + \underbrace{(\vec{y}, \vec{y})}_{\Downarrow\ 0}$$

$$\nabla_{\vec{x}} (A\vec{x}, A\vec{x}) = ?$$

$$\frac{d}{dx}[x \cdot x] \quad \text{ordinary scalar variables}$$

$$= \frac{d}{dx}(x) \cdot x + x \frac{d}{dx}(x)$$

So $\nabla_x (A\vec{x}, A\vec{x}) =$

$$\nabla_{\vec{x}} (\vec{x}, A^T A\vec{x}) + \nabla_x (A^T A\vec{x}, \vec{x})$$

$$= 2A^T A\vec{x}$$

$$\nabla_{\vec{x}} \|A\vec{x}-\vec{y}\|^2 =$$

$$\nabla_x \left\{ (A\vec{x}, A\vec{x}) - (\vec{y}, Ax) - (Ax, y) + (\vec{y}, \vec{y}) \right\}$$

$$= 2A^T A\vec{x} - A^T y - A^T y + 0$$

So if we set $\nabla_x \|A\vec{x}-\vec{y}\|^2 = 0$

$$\Rightarrow \boxed{A^T A \vec{x} = A^T \vec{y}}$$ normal Equation

<u>Notes:</u> $A^T A$ is always square and symmetric

$$A \in \mathbb{R}^{n \times m} \Rightarrow A^T \in \mathbb{R}^{m \times n}$$

So $A^T A$ involves

sum over n $\quad m \times n \cdot n \times m$

$$\Rightarrow A^T A \in R^{m \times m}$$

and
$$(A^T A)_{ij} = \sum_{k=1}^{n} A^T_{ik} A_{kj}$$
$$= \sum_k A_{ki} A^T_{jk}$$
$$= \sum A^T_{jk} A_{ki}$$
$$= (A^T A)_{ji}$$

So $A^T A$ is symmetric.

Summary on Least squares

if $A\vec{x}=\vec{y}$ has no solution

we can seek a generalized

solution:

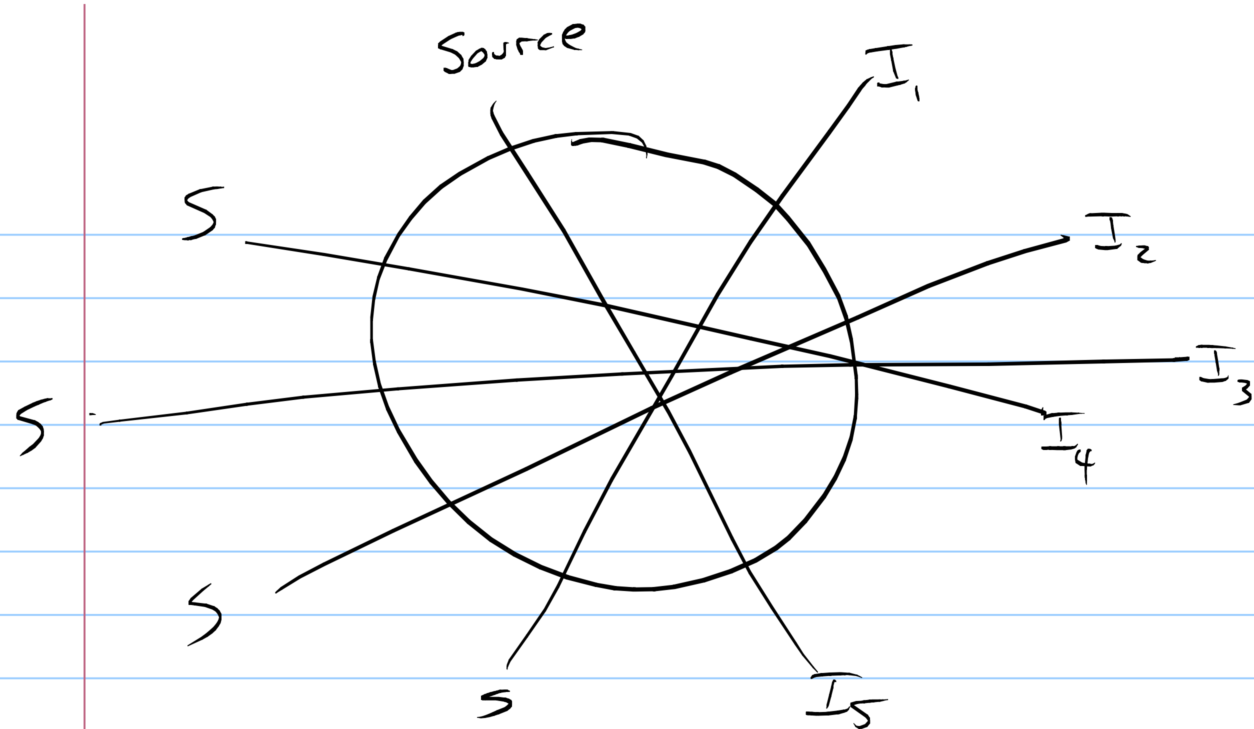
$$\min_{\vec{x}} \|A\vec{x}-\vec{y}\|^2$$

$$\Longrightarrow \quad A^T A \vec{x} = A^T Y$$

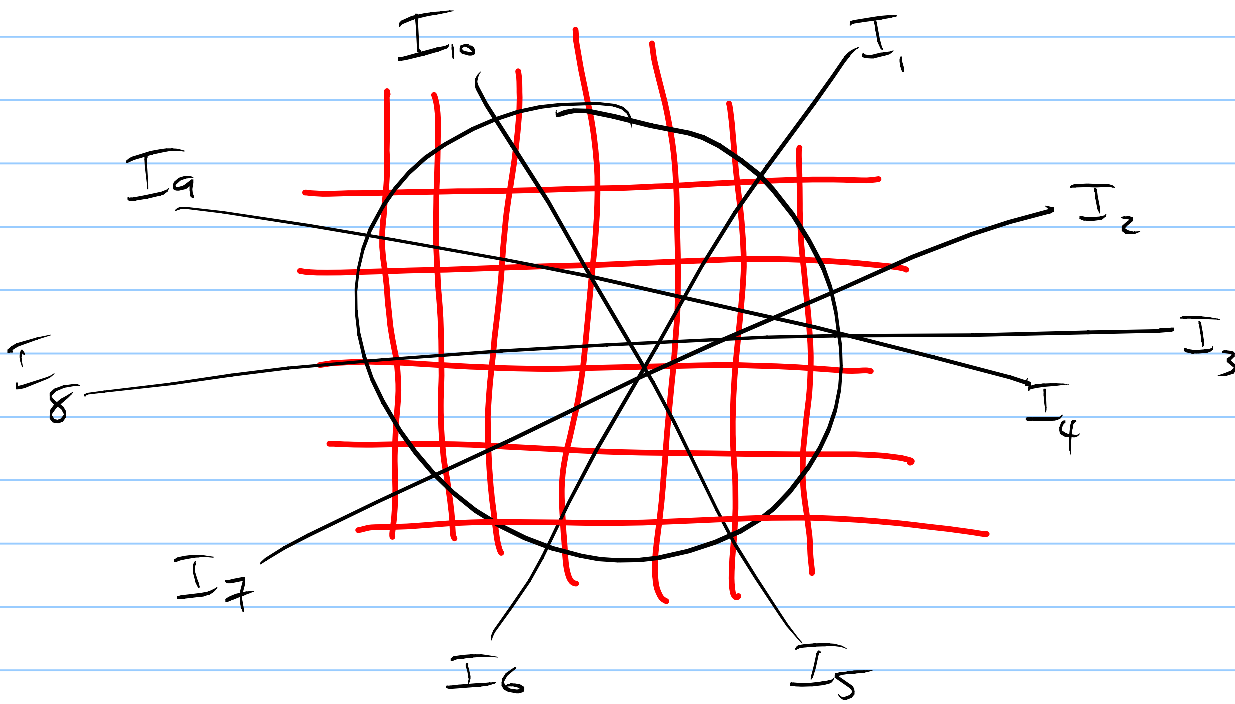
True for any A.

more real examples

- GPS triangulation
- x-ray tomography
  CAT Scan



Grid the target



M cells   N data

Each msmt corresponds to 1 row of a matrix

Each parameter value (cell) correspond to 1 column

$$J \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}$$

absorption

measured x-ray intensity

$$J \in R^{m \times n}$$

Solve $\min_{\vec{a}} \| J\vec{a} - \vec{I} \|^2$

$\boxed{\text{Tomography}}$

Eigenvalues / Eigenvectors

Consider the action of a generic matrix on a vector:

$$A\vec{x} \qquad A \in \mathbb{R}^{n \times n} \quad \vec{x} \in \mathbb{R}^{n}$$

normally this will change direction and length of $\vec{x}$

suppose

$$A\vec{x} = \alpha \vec{x}$$

$\alpha \downarrow$ scalar

So A maps $\vec{x}$ into a scalar multiple of itself.

$$A\vec{x} = \alpha \vec{x} \quad \star$$

$$\Rightarrow A\vec{x} = \alpha I \vec{x}$$

I: $n \times n$ identity matrix

$$I \cdot \vec{x} = \vec{x}$$

$$\Rightarrow (A - \alpha I)\vec{x} = 0$$

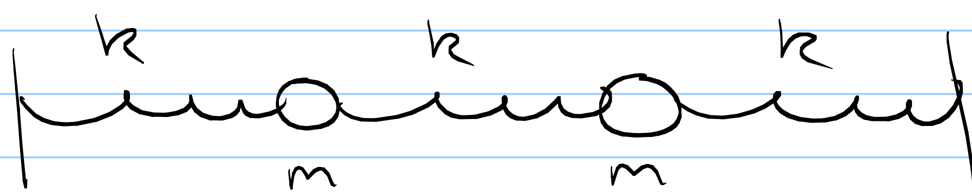
So, for $\star$ to be true

$\vec{x}$ must be in the null space of $\underline{A - \alpha I}$

if $A - \alpha I$ has a null space, it cannot be invertible. So,

$$\boxed{\operatorname{Det}(A - \alpha I) = 0}$$

Example



Can show

$$\ddot{x}_1 = -2\omega_0^2 x_1 + \omega_0^2 x_2$$
$$\ddot{x}_2 = -2\omega_0^2 x_2 + \omega_0^2 x_1$$

Seek solutions: $x_1 = Ae^{i\omega t}$
$x_2 = Be^{i\omega t}$

Plug into ☆

$$\underbrace{\begin{pmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{pmatrix}}_{\text{matrix}} \underbrace{\begin{pmatrix} A \\ B \end{pmatrix}}_{\text{vect}} = \underbrace{\omega^2}_{\text{scalar}} \underbrace{\begin{pmatrix} A \\ B \end{pmatrix}}_{\text{vector}}$$

$$K \vec{u} = \omega^2 \vec{u}$$

$$\bar{u} = \begin{pmatrix} A \\ B \end{pmatrix} \qquad K = \begin{pmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{pmatrix}$$

Examples

$$A = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$$

$$AX = \lambda I X$$

$$\Rightarrow (A - \lambda I) X = 0$$

$$\Rightarrow \operatorname{Det}(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{pmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{pmatrix}$$

$$\operatorname{Det}(A - \lambda I) =$$

$$(5-\lambda)^2 - 1 = 0$$

$$(5-\lambda) = \pm 1$$

$$5 \mp 1 = \lambda$$

So $\underline{\lambda = 6 \text{ or } 4}$

Now for the ε-vectors

$$\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 6\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \quad 5x + y = 6x$$
$$x + 5y = 6y$$

$x = 1, \ y = 1$ $\quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eig. vector

$$\boxed{\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$6, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvalue eigenvector pair

Next

$$\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$$

$$5x + y = 4x$$
$$x + 5y = 4y$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So $4, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the second ε-value/vector pair.