MATH 348 - Advanced Engineering Mathematics July 15, 2009

Homework 8, Summer 2009 **Due**: July 24, 2009

Fourier Transforms

1. Calculate the following Fourier sine/cosine transformations. Please include the domain which the transformation is valid.
   (a) $\mathfrak{F}_c(e^{-ax}),\ \ a \in \mathbb{R}^+$
   (b) $\mathfrak{F}_c^{-1}\left(\frac{1}{1+\omega^2}\right)$
   (c) $\mathfrak{F}_s(e^{-ax}),\ \ a \in \mathbb{R}^+$
   (d) $\mathfrak{F}_s^{-1}\left(\sqrt{\frac{2}{\pi}}\frac{\omega}{a^2+\omega^2}\right),\ \ a \in \mathbb{R}^+$

2. Calculate the following transforms:
   (a) $\mathfrak{F}\{f\}$ where $f(x) = \delta(x - x_0),\ \ x_0 \in \mathbb{R}$.[1]
   (b) $\mathfrak{F}\{f\}$ where $f(x) = e^{-k_0|x|},\ \ k_0 \in \mathbb{R}^+$.
   (c) $\mathfrak{F}^{-1}\left\{\hat{f}\right\}$ where $\hat{f}(\omega) = \frac{1}{2}\left(\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right),\ \ \omega_0 \in \mathbb{R}$.
   (d) $\mathfrak{F}^{-1}\left\{\hat{f}\right\}$ where $\hat{f}(\omega) = \frac{1}{2}\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right),\ \ \omega_0 \in \mathbb{R}$.
   (e) Find $\hat{f}(\omega)$ where $f(x+c)$, $c \in \mathbb{R}$.

3. The convolution h of two functions f and g is defined as[2],

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)dp = \int_{-\infty}^{\infty} f(x-p)g(p)dp. \tag{1}$$

   (a) Show that $\mathfrak{F}\{f * g\} = \sqrt{2\pi}\mathfrak{F}\{f\}\mathfrak{F}\{g\}$.
   (b) Find the convolution $h(x) = (f * g)(x)$ where $f(x) = \delta(x - x_0)$ and $g(x) = e^{-x}$.

4. Given the ODE,

$$y' + y = f(x), \quad -\infty < x < \infty. \tag{2}$$

   Let $f(x) = \delta(x)$ and then:
   (a) Calculate the frequency response associated with (2). [3]
   (b) Calculate the Green's function associated with (2).
   (c) Using convolution find the steady-state solution to the (2).

5. List three questions you have associated with Fourier series and three questions you have associated with Fourier transforms and submit them as the leading page to this homework assignment.[4]

---

[1] Here the δ is the so-called Dirac, or continuous, delta function. This isn't a function in the true sense of the term but instead what is called a generalized function. The details are unimportant and in this case we care only that this Dirac-delta *function* has the property $\int_{-\infty}^{\infty}\delta(x-x_0)f(x)dx = f(x_0)$. For more information on this matter consider `http://en.wikipedia.org/wiki/Dirac_delta_function`. To drive home that this *function* is strange, let me spoil the punch-line. The sampling function $f(x) = \text{sinc}(ax)$ can be used as a definition for the Delta *function* as $a \to 0$. So can a bell-curve probability distribution. Yikes!

[2] Here wee keep the same notation as Kreysig pg. 523

[3] this is often called the steady-state transfer function

[4] I will write up a Q+A sheet addressing both large and shared misunderstandings associated with our sections questions and post them on the ticc website.