Advanced Engineering Mathematics		**Homework Two**
Introduction to Linear Vectors Spaces : Span, Linear Independence, Basis, Dimension		
Text: 7.4, 7.9	Lecture Notes: 5-6	Lecture Slide: 2

Quote of Homework Two Solutions
Grandpa Joe: But this roof is made of glass, it'll shatter into a thousand pieces. We'll be cut to ribbons! **Willy Wonka**: Probably.
Roald Dahl : Willy Wonka and the Chocolate Factory (1971)

1. Vocabulary of Vector Spaces

Given,

$$\mathbf{A}_1 = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix},$$

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix},$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

Before we get into these problems we record the following row-reductions:

(1)
$$[\mathbf{A}_1\,|\mathbf{b}_1\,] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

(2)
$$\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ -3 & 8 & h \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2h+20 & 0 \end{array}\right]$$

(3)
$$\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3] = \begin{bmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & h-10 \end{bmatrix}$$

(4)
$$[\mathbf{X}|\mathbf{y}] = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3|\mathbf{y}] = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(5)
$$[\mathbf{A}_2|\mathbf{b}_2] \sim \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 0 & 20 & 10 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

1.1. **Linear Combinations.** Is $\mathbf{b}_1$ a linear combination of the columns of $\mathbf{A}_1$?

Recall the the following equivalence for $\mathbf{A} \in \mathbb{R}^{m\times n}$,

$$\mathbf{A}\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{a}_i, \tag{6}$$

which says that the matrix product is the same as a linear combination of its columns. So, asking if a vector, $\mathbf{b}$, is a linear combination of columns is also asking if $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution. Either way we study the pivot structure of $[\mathbf{A}|\mathbf{b}]$. Thus, from above we have that $\mathbf{A}_1\mathbf{x} = \mathbf{b}_1$ has no solution and therefore $\mathbf{b}_1$ cannot be written as a linear combination of the columns from $\mathbf{A}$.

1.2. **Linear Dependence.** Determine all values for h such that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a linearly dependent set.

A set of n-many vectors, $\mathbf{v}_i$, forms a linearly independent set if and only if $c_i = 0$ for $i = 1, 2, 3, \ldots, n$ is the only solution to $\sum_{i=1}^{n} c_i\mathbf{v}_i = \mathbf{0}$. This is equivalent to asking if $\mathbf{V}\mathbf{c} = \mathbf{0}$ has only the trivial solution, where $\mathbf{V}$ is a matrix formed by the set of vectors. If a homogeneous system has the trivial solution then there must be a pivot for every variable. If we want the vectors to form a linearly **de**pendent set then we must have the existence of free variables. Thus, from above, we require $h = -10$.

1.3. **Linear Independence.** Determine all values for h such that $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ forms a linearly independent set.

We repeat the argument of 1.2 and now require a pivot for each variable. In this case we have no values of h such that $c_1 = c_2 = c_3 = 0$ is the only solution to $\sum_{i=1}^{3} c_i\mathbf{w}_i = 0$. Thus, the vectors ALWAYS form a linearly dependent set.

1.4. **Spanning Sets.** How many vectors are in $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$? How many vectors are in span(S)? Is $\mathbf{y} \in$span(S)?

This is all a question of language. The set S has three elements. However, the spanning set of S is the set of all linear combinations of the vectors in S. That is, the spanning set of S is every vector that takes the form, $\sum_{i=1}^{3} c_i\mathbf{x}_i$ for any $c_i \in \mathbb{R}$. This spanning set, by definition has infinitely many elements.[1] Finally, we ask if $\mathbf{y}$ is in this spanning set, which is really asking if there are c_i's such that $\mathbf{y}$ can be written as $\sum_{i=1}^{3} c_i\mathbf{x}_i$. Again, this is the same as asking if $\mathbf{X}\mathbf{c} = \mathbf{y}$, which is addressed by understanding the solubility of $[\mathbf{X}|\mathbf{y}]$. From the previous row-reductions we see that this system has a solution, in fact is has many, and therefore $\mathbf{y} \in \text{span}(S)$.

1.5. **Matrix Spaces.** Is $\mathbf{b}_2 \in \text{Nul}(\mathbf{A}_2)$? Is $\mathbf{b}_2 \in \text{Col}(\mathbf{A}_2)$?

Recall that the null-space of a matrix is the set of all solutions to $\mathbf{A}\mathbf{x} = \mathbf{0}$. This space tells us about all the points in space the homogeneous linear equations simultaneously intersect. One way to determine if $\mathbf{b}_2$ is in the null-space of $\mathbf{A}_2$ is by solving the homogeneous equation and determining if $\mathbf{b}_2$ is one of these solutions. However, it pays to note that if $\mathbf{b}_2$ is in the null-space of $\mathbf{A}_2$ then $\mathbf{A}_2\mathbf{b}_2 = \mathbf{0}$. A quick check shows,

$$\mathbf{A}_2\mathbf{b}_2 = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \mathbf{0}. \tag{7}$$

The column space, on the other hand, is a little different. The column space is the set of all linear combinations of the columns of $\mathbf{A}_2$. This is also called the spanning set of the columns of $\mathbf{A}_2$. Thus, this question can be addressed in the same way as problem 1.1 or the last part of 1.4 and we have,

$$[\mathbf{A}_2|\mathbf{b}_2] \sim \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 0 & 20 & 10 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{8}$$

The conclusion is that $\mathbf{b}_2$ is in both the null-space and column space of $\mathbf{A}_2$. This is not generally true of a non-trivial vector. In fact, it is never true for rectangular coefficient data.

[1] As a simple case consider every scaling of $\hat{\mathbf{i}}$ how many elements would be in this set?

Given,

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}.$$

The following problems will require the use of an echelon form of **A**. One such form is,

(9)
$$\mathbf{A} \sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{B}.$$

2.1. **Null Space.** Determine a basis and the dimension of Nul(**A**).

The null-space of **A** is the set of all solutions to $\mathbf{Ax} = \mathbf{0}$. To find a basis for this space we must explicitly solve the homogeneous equation. Thus, from the echelon form **B** we have the following,

$$\begin{aligned} x_4 &= -3x_5 \\ x_3 &= (x_4 - x_5)/3 = (-3x_5 - x_5)/3 = -\frac{4}{3}x_5 \\ x_1 &= \frac{1}{2}(3x_2 - 6x_3 - 2x_4 - 5x_5) = \frac{1}{2}(3x_2 - 6(-\frac{4}{3}x_5) - 2(-3x_5) - 5x_5) = \\ &= \frac{1}{2}(3x_2 + 8x_5 + 6x_5 - 5x_5) = \frac{3}{2}x_2 + \frac{9}{2}x_5 \\ x_2 &\in \mathbb{R} \\ x_5 &\in \mathbb{R} \\ &\Rightarrow \mathbf{x} = x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \qquad x_2, x_5 \in \mathbb{R} \end{aligned}$$

Hence, the basis for Nul(**A**) is

(10)
$$B_{null} = \left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \right\}$$

and dim(Nul **A**) = 2. The conclusion is that the five linear four-dimensional objects intersect at many points in $\mathbb{R}^5$. The collection of points forms a two-dimensional subspace, which is spanned by the basis vectors. That is, the linear objects intersect forming a planer subspace of $\mathbb{R}^5$. [2]

2.2. **Column Space.** Determine a basis and the dimension of Col(**A**).
The column-space is the set of all linear combinations of the columns of **A**. We would like to know a basis for this space, which implies that we must somehow determine the columns of the **A** matrix that contain unique directional information. That is, we must find the linearly independent columns of the **A** matrix. This information has been made clear through the previous null-space problem. Recall that if a set of vectors is linearly independent then their corresponding homogeneous equation must only have the trivial solution. Since row-reduction does not change the solution to a homogeneous equation we have,

(11)
$$\mathbf{Ax} = \mathbf{0} \iff [\mathbf{A}|\mathbf{0}] \sim [\mathbf{B}|\mathbf{0}] \iff \mathbf{Bx} = \mathbf{0}.$$

So, we can see if the columns of **A** are linearly independent by considering the linear independence of the columns of **B**. Clearly, the previous problem shows that the columns of **B** are not linearly independent. However, it is also clear from **B** that the columns without pivots can be made using the columns with pivots, $\mathbf{b}_2 = -3/2\mathbf{b}_1$ and $\mathbf{b}_5 = 3\mathbf{b}_4 + 4/3\mathbf{b}_3 - (9/2)\mathbf{b}_1$. So, if we take only the pivot columns

[2] It is important to notice how the dimension of the null-space drives the previous statements. I did not draw or try to picture anything.

from $\mathbf{B}$ then we would loose the linearly dependent columns and their free-variables. Consequently, the only solution to $\mathbf{B}_{change} = \mathbf{0}$ would be the trivial solution, which implies the columns are linearly independent.

There is still a problem. While row-reduction did not change the dependence relation, it did change the actual vectors. That is, the column-space of $\mathbf{A}$ is different the the column-space of $\mathbf{B}$. To see this consider the constants neceesary for $\mathbf{a}_1 \stackrel{?}{=} c_1\mathbf{b}_1 + c_2\mathbf{b}_3 + c_3\mathbf{b}_4$. [3] So, the conclusion is that we must take the linearly independent columns from $\mathbf{A}$ as told to us by $\mathbf{B}$. Thus, a basis for the column space of $\mathbf{A}$ are the pivot columns of $\mathbf{A}$,

$$B_{ColA} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \right\} \tag{12}$$

and $\dim(\text{Col}\mathbf{A}) = 3$. The dimension of the column-space is also known as the rank of $\mathbf{A}$. From this we see an example of the so-called rank-nullity theorem, which says that the dimension of the null-space and the dimension of the column-space must always add to be the total number of columns in the matrix. That is,

$$\text{Rank } \mathbf{A} + \dim(\text{Nul}(\mathbf{A})) = n, \text{ where } \mathbf{A} \in \mathbb{R}^{m \times n}.$$

2.3. **Row Space.** Determine a basis and the dimension of Row $\mathbf{A}$. What is the Rank of $\mathbf{A}$?

The row-space of a matrix is the set of all linear combinations of its rows. A basis can be found by taking only the linearly independent rows of the matrix, which can be clearly seen as the non-zero rows of any echelon form. While in the case of a column-space the columns must necessarily come from the original matrix, this is not a requirement for the row-space.[4] Thus, a basis for the row-space of $\mathbf{A}$ is given by,

$$B_{RowA} = \left\{ \begin{matrix} [2\ -3\,6\,2\,5] \\ [0\,0\,3\ -1\,1] \\ [0\,0\,0\,1\,3] \end{matrix} \right\}. \tag{13}$$

Since these rows were chosen because of their pivots, the dimension of this space is always equal to the dimension of the column space and $\dim(\text{Row}\mathbf{A}) = \text{Rank } \mathbf{A} = 3$. [5]

3. Abstract Vector Spaces - Function Spaces

Given,

$$\left[m\frac{d^2}{dt^2} + k\right] y = 0, \;\; m, k \in \mathbb{R}^+ \tag{14}$$

$$\left[\frac{d}{dt} - \mathbf{A}\right] \mathbf{Y} = 0, \;\; \mathbf{A} \in \mathbb{R}^{2\times 2} \tag{15}$$

3.1. **Equivalence of Equations.** Find the change of variables that maps (14) onto (15) and using this define $\mathbf{Y}$ and $\mathbf{A}$.
Consider the variable transformation defined by, $y' = v$. In this case, (14) becomes,

$$\frac{dv}{dt} = -\frac{k}{m}y, \tag{16}$$

and coupled with the transformation we have,

$$\frac{dy}{dt} = v, \tag{17}$$

$$\frac{dv}{dt} = -\frac{k}{m}y, \tag{18}$$

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[3] Answer : There are no constants that allow for this to be true.

[4] The reason for this is that row-operations <u>are</u> linear combinations, $R_i = R_i + \alpha R_j$. Thus using the non-zero rows of any echelon form you can get back to the rows of the original matrix and all linear combinations for that matter.

[5] It is possible to take the corresponding rows from $\mathbf{A}$ but dangerous. The reason why is that the rows of the echelon form may not correspond directly to the rows of the original matrix because of row-swaps. However, if you wanted to take the rows from $\mathbf{A}$ and have kept track of your row-swaps then there shouldn't be a problem.

which is a linear system of differential equations where $\mathbf{Y}(t) = [y(t)\ v(t)]^{\mathrm{T}}$ and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}. \tag{19}$$

3.2. **Function Spaces.** Find the general solution to (15) and for $m = k = 1$ sketch its associated real phase-portrait.

In this case we have that $\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = \lambda^2 + 1 = 0$. The eigenvalues are then $\lambda = \pm i$ and the eigenvectors are $\mathbf{Y} = [1\ \pm i]^{\mathrm{T}}$ and the general solution in real form is,

$$\mathbf{Y}(t) = c_1 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{20}$$

which is a clockwise rotation of the vector $\mathbf{c} = [c_1\ c_2]^{\mathrm{T}}$. Thus the phase-portrait is a clockwise parametrization of a circle of length $\sqrt{\mathbf{c}^{\mathrm{T}}\mathbf{c}}$.

4. Introduction to Infinite Dimensional Spaces

Given,

$$y'' + \lambda y = 0, \ \ \lambda \in \mathbb{R} \tag{21}$$

$$y(0) = 0, \ \ y(\pi) = 0. \tag{22}$$

4.1. **General Solution.** Find the general solution to the ODE (21).

To solve this problem we make the assumption that $y(t) = e^{rt}$ to get that,

$$y'' + \lambda y = r^2 e^{rx} + \lambda e^{rx} = e^{rx}\left(r^2 + \lambda\right) = 0, \tag{23}$$

which has roots $r = \mp\sqrt{-\lambda}$. This leads to the following general solutions, which depend on the value of λ,

$$\lambda > 0 : y_1(x) = b_1 e^{i\sqrt{\lambda}x} + b_2 e^{-i\sqrt{\lambda}x}, \tag{24}$$

$$\lambda < 0 : y_2(x) = b_3 e^{\sqrt{|\lambda|}x} + b_4 e^{-\sqrt{|\lambda|}x}, \tag{25}$$

$$\lambda = 0 : y_3(x) = b_5 e^{0\cdot x} + b_6 x e^{0\cdot x}. \tag{26}$$

Often it is easier to express these solutions in terms of the *trigonometric* and *hyperbolic trigonometric* functions. Doing so gives,

$$\lambda > 0 : y_1(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), \tag{27}$$

$$\lambda < 0 : y_2(x) = c_3 \cosh\left(\sqrt{|\lambda|}x\right) + c_4 \sinh\left(\sqrt{|\lambda|}x\right), \tag{28}$$

$$\lambda = 0 : y_3(x) = b_5 + b_6 x. \tag{29}$$

These are the general solutions to (21) based on the value of λ.

4.2. ***Boundary Value Problem.*** Show that the nontrivial solutions to (21) that satisfy the associated boundary conditions (22) requires that $\lambda = n^2 \in \mathbb{N}$.

Not all of the solutions outlined in (27)-(29) will also satisfy (22). A quick geometric argument implies that $b_6 = b_5 = c_4 = c_3 = c_1 = 0$; the value of c_2 is inconsequential. [6] Now, of these sine functions only those of specific angular frequency will satisfy both conditions in (22). To see this consider that the right boundary condition implies,

$$y_1(\pi) = c_2 \sin\left(\sqrt{\lambda}\pi\right) = 0, \tag{30}$$

which can be satisfied if $c_2 = 0$ or $\sin(\cdot) = 0$. The former would give us only trivial solutions, while the latter implies that $\sqrt{\lambda}\pi = n\pi$ for $n = 1, 2, 3, \ldots$. Hence, $\lambda = n^2$ and the solutions to (21) that satisfy (22) are $y_n(x) = \sin\left(\sqrt{\lambda_n}x\right)$ where $\lambda_n = n^2$. This problem is a classic example of a so-called Sturm-Liouville problem and the λ_n, y_n are called eigenvalue/eigenfunction pairs.

[6] The argument works like this. First, (29) is a line and (22) requires that this line pass through the origin and the point $(\pi, 0)$. The only line that does this is the line $y(x) = 0$. Next, (28) are hyperbolic trigonometric functions whose graphs are given on wikipedia. The point is that $\cosh(x)$ is never zero and $\sinh(x)$ is zero only at the origin. Thus the only way for this function to satisfy both conditions is trivially. Lastly, of the two trigonometric functions in (27) only sine passes through the origin. Thus, we keep the sine function and discard the rest.

4.3. **Infinite Dimensional Space.** Show that $u_n(x,t) = \sin(nx)e^{-n^2t}$ satisfies $[\partial_t - \partial_{xx}]\, u = 0$.

The study of infinite dimensional spaces, typically, begins with a partial differential equation, which can be thought of as an equation that contains infinitely-many ODEs. Later we will see why this is the case but for now we consider the set of solutions to $[\partial_t - \partial_{xx}]\, u = 0$ defined by $u_n(x,t) = \sin(nx)e^{-n^2t}$ for $n = 1, 2, 3, \ldots$. Each u_n solves the PDE and its spatial component, $\sin(nx)$, manifests from the previous boundary value problem. To check this solution we substitute u_n into the PDE to get,

$$[\partial_t - \partial_{xx}]\, u_n = [\partial_t - \partial_{xx}] \sin(nx)e^{-n^2t} \tag{31}$$

$$= -n^2 \sin(nx)e^{-n^2t} + n^2 \sin(nx)e^{-n^2t} \tag{32}$$

$$= 0, \tag{33}$$

for $n = 1, 2, 3, \ldots$. Now, here is the interesting consequence. Since the PDE is linear, an arbitrary linear combination of solutions is also a solution. Thus, a general solution to this PDE is given by,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)e^{-n^2t}, \tag{34}$$

which, we notice, is an infinite sum of basis vectors and we conclude that the solution space of the PDE is an infinite dimensional space with basis vectors $\sin(nx)e^{-n^2t}$. While this is a useful perspective it is also important to study the qualities of such a solution. However, to do this one must study so-called Fourier series.

5. *Hilbert Spaces*

Given,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{35}$$

5.1. **General Solutions to the *Heat Equation*.** Justify that $u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)e^{-n^2t}$ is a solution to (35).

Notice that (35) can be rewritten to be,

$$\left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right] u = 0, \tag{36}$$

which is in the form of 4.3. In 4.3 it was shown that $u_n(x,t) = \sin(nx)e^{-n^2t}$ solves this linear partial differential equation (PDE).[7] It should be clear from your study of ODE's and our study of linear systems that the linear combination of solutions to a linear problem is again a solution to the same problem. Thus, $u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)e^{-n^2t}$ must also solve the PDE. [8]

5.2. **Abstract Inner Products.** Show that $\langle f, g\rangle = (f,g) = \int_{-\pi}^{\pi} f(x)g(x)\, dx$ satisfies the three axioms of a *Real Inner Product Space* found on page 326.

First, we must agree that the space of integrable functions, V, satisfies the algebra/axioms of vector spaces, page 324, where $f(x) = 0$ and $g(x) = 1$ are the additive and multiplicative identities, respectively. With that said we now hope that,

$$\langle f, g\rangle = (f,g) = \int_{-\pi}^{\pi} f(x)g(x)\, dx, \tag{37}$$

satisfies the axioms of a real inner-product space. Checking the three axioms for $c_1, c_2 \in \mathbb{R}$ and $f, g, h \in V$ we have,

(1) Axiom I : Linearity

$$\langle c_1 f + c_2 g, h\rangle = \int_{-\pi}^{\pi} [c_1 f(x) + c_2 g(x)]\, h(x)\, dx \tag{38}$$

$$= c_1 \int_{-\pi}^{\pi} f(x)h(x)\, dx + c_2 \int_{-\pi}^{\pi} g(x)h(x)\, dx \tag{39}$$

$$= c_1 \langle f, h\rangle + c_2 \langle g, h\rangle \tag{40}$$

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[7]The equation is linear since it is a linear combination of derivatives of the unknown function u.

[8]If you don't believe this argument then substitute this solution into the PDE and, without worrying about swapping the order of derivative and infinite-summation, show that equality is maintained.

(2) Axiom II : Symmetry

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx \tag{41}$$

$$= \int_{-\pi}^{\pi} g(x)f(x)\,dx \tag{42}$$

$$= \langle g,f\rangle \tag{43}$$

(3) Axiom III : Positive Definiteness
The following inner-product,

$$\langle f,f\rangle = \int_{-\pi}^{\pi} f(x)\cdot f(x)\,dx \tag{44}$$

$$= \int_{-\pi}^{\pi} [f(x)]^2\,dx, \tag{45}$$

is the integral of a non-negative function and implies that $\langle f,f\rangle \geq 0$. Since $f(x)=0$ is the only function whose square contains zero area under its curve we also conclude that $\langle f,f\rangle = 0 \iff f(x)=0$.

The previous results show that the space of integrable functions is also a real inner-product space with inner-product defined as the previous integral.

5.3. **Orthogonality Relation.** Show that $\langle \sin(nx),\sin(mx)\rangle = \pi\delta_{nm}$ for all $n,m\in\mathbb{N}$.
This integral will be very important throughout the study of Fourier series and PDE and is an archetype of an orthogonality argument for abstract real inner-product spaces. The result listed above is what we will use but first we must justify the equality. To do we we consider the following argument for $n,m\in\mathbb{Z}$,

$$\left\langle e^{inx}, e^{\pm imx}\right\rangle = \int_{-\pi}^{\pi} e^{inx}e^{\pm imx}\,dx \tag{46}$$

$$= \int_{-\pi}^{\pi} e^{i(n\pm m)x}\,dx \tag{47}$$

$$= (n\pm m)^{-1}\, e^{i(n\pm m)x}\Big|_{-\pi}^{\pi} \tag{48}$$

$$= (n\pm m)^{-1}\left[e^{i(n\pm m)\pi} - e^{-i(n\pm m)\pi}\right] \tag{49}$$

$$= (n\pm m)^{-1}\left[(-1)^{n\pm m} - (-1)^{n\pm m}\right] \tag{50}$$

$$= \begin{cases} 0, & \text{for } e^{imx} \text{ and any } n,m \\ 0, & \text{for } e^{-imx} \text{ and } n\neq m \end{cases}. \tag{51}$$

To relate this to the integral in question we note that,

$$\text{Real}\left[\left\langle e^{inx}, e^{\pm imx}\right\rangle\right] = \int_{-\pi}^{\pi} [\cos(nx)\cos(mx) \mp \sin(nx)\sin(mx)]\,dx = 0 \Rightarrow \int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx = \pm\int_{-\pi}^{\pi}\sin(nx)\sin(mx)\,dx, \tag{52}$$

which implies that each integral is zero. [9] When $n=m$ it is quick to verify $\left\langle e^{inx}, e^{-inx}\right\rangle = 2\pi$. Using the previous relations we have,

$$\text{Real}\left[\left\langle e^{inx}, e^{-inx}\right\rangle\right] = \int_{-\pi}^{\pi} [\cos(nx)\cos(nx) + \sin(nx)\sin(nx)]\,dx \tag{53}$$

$$= 2\int_{-\pi}^{\pi}\sin(nx)\sin(nx)\,dx \tag{54}$$

$$= 2\pi. \tag{55}$$

Taken together we have the desired result, $\langle \sin(nx),\sin(mx)\rangle = \pi\delta_{nm}$.

5.4. **Initial Conditions and Unknown Constants.** Suppose the solution to (35) is initially known to be $u(x,0)=f(x)$. Using the previous orthogonality relations, show that $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$.

First, note that the initial condition implies,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx). \tag{56}$$

[9]The only number such that $A=\pm A$ is true is the number zero.

If we apply the linearity of the inner-product and the orthogonality relation we get,

$$\langle f(x), \sin(mx) \rangle = \left\langle \sum_{n=1}^{\infty} b_n \sin(nx), \sin(mx) \right\rangle \tag{57}$$

$$= \sum_{n=1}^{\infty} b_n \langle \sin(nx), \sin(mx) \rangle \tag{58}$$

$$= \sum_{n=1}^{\infty} b_n \pi \delta_{nm} \tag{59}$$

$$= \pi b_m, \tag{60}$$

which implies that $b_m = \frac{1}{\pi} \langle f(x), \sin(mx) \rangle$. Since m is arbitrary we can formally change it to n and rewriting the inner-product as its integral gives the desired result.