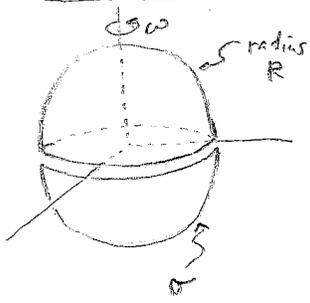


### Problem 8.3

Let's find the force that acts on the northern hemisphere.



The general expression for the net force acting on the charges in some volume ( $V$ ) bounded by a surface ( $S$ ) is given by:

$$\vec{F} = \oint_S \vec{T} \cdot d\vec{a} - \mu_0 \epsilon_0 \frac{d}{dt} \int_V \vec{S} d\tau$$

In this case  $\vec{E} \propto \vec{B}$  are constant  $\Rightarrow \vec{S}$  is constant, so the volume integral will be time independent so the second term will be zero. So now:

$$\vec{F} = \oint_S \vec{T} \cdot d\vec{a}$$

So we need to integrate the stress tensor ( $\vec{T}$ ) over some surface that encloses the Northern hemisphere of the spherical shell.

I will show how to do it for two different surfaces:

$S_1$  = The entire x-y plane cuts the shell in half. We still have to close the surface though, so we will close it by a half-sphere out at infinite radius (above the x-y plane). You can visualize the surface by imagining you have the sketch shown above and you take the radius of the surface to infinity. This surface encloses a lot more than just the northern hemisphere of the shell (in fact, it includes all of space for  $z \geq 0$ ), but the only charged stuff inside is the northern hemisphere of the shell, so this surface will work.

$S_2$  = a disk of radius slightly larger than  $R$  in the x-y plane, and then a hemisphere with radius slightly larger than  $R$  above the x-y plane that blankets (covers) the northern hemisphere of the spherical shell. Technically these both have radius =  $R + \epsilon$ , but we will take the limit as  $\epsilon \rightarrow 0$  to get the surface to match the spherical shell, so we might as well just use the radius as  $R$ . The only time you need to worry about this is if the fields are discontinuous across the boundary (which is not the case for the  $\vec{B}$ -field).

To calculate the stress tensor, we need to know the fields:

$$\vec{B}_{\text{inside}} = \frac{2}{3} \mu_0 \sigma R \omega \hat{z} \quad (\text{constant field})$$

$$\vec{B}_{\text{outside}} = \frac{\mu_0}{3} (\sigma R \omega) \left(\frac{R}{r}\right)^3 \left[ 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right] \quad (\text{magnetic dipole field})$$

The magnetic part of the stress tensor is given by:

$$(T_{ij})_{\text{mag}} = \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)$$

By symmetry, we can see that the net force must be in the  $\hat{z}$  direction, so we only need the  $z$  component of the force equation

$$(\vec{T} \cdot d\vec{a})_z = T_{zx} da_x + T_{zy} da_y + T_{zz} da_z$$

So for  $S_1$ , we would have

$$F_z = \oint_{S_1} (\vec{T} \cdot d\vec{a})_z = \int_{\substack{\text{disk} \\ \text{of radius} \\ R}} (\vec{T} \cdot d\vec{a})_z + \int_{\substack{\text{rest of} \\ \text{x-y plane}}} (\vec{T} \cdot d\vec{a})_z + \int_{\substack{\text{hemisphere} \\ \text{at} \\ \text{infinity}}} (\vec{T} \cdot d\vec{a})_z$$

The reason the  $x$ - $y$  plane has to be broken up is because there is a different expression for  $\vec{B}$  inside and outside the shell.

Looking at the hemisphere at infinity:

$$\vec{T} \sim B^2 \sim \frac{1}{r^6} \quad \Rightarrow \quad \vec{T} \cdot d\vec{a} \sim \frac{1}{r^4} \quad \text{and as } r \rightarrow \infty, \text{ the integrand} \rightarrow 0$$

$$d\vec{a} \sim r^2$$

So the hemisphere at infinity will give no contribution

This now gives

$$F_z = \oint_{S_1} (\vec{T} \cdot d\vec{a})_z = \int_{\substack{\text{disk} \\ \text{of radius} \\ R}} (\vec{T} \cdot d\vec{a})_z + \int_{\substack{\text{rest of} \\ \text{x-y plane}}} (\vec{T} \cdot d\vec{a})_z$$

Lets find the force on the disk:

Disk

$$(\vec{T} \cdot d\vec{a})_z = T_{zx} da_x + T_{zy} da_y + T_{zz} da_z$$

For this surface  $da_x = da_y = 0 \Rightarrow$  we only need  $T_{zz}$  for the disk.

$$T_{zz} = \frac{1}{\mu_0} (B_z B_z - \frac{1}{2} B^2)$$

The disk is inside the shell, so  $B$  is constant and  $B = B_z$

$$T_{zz} = \frac{1}{2\mu_0} B^2 = \frac{1}{2\mu_0} 4\mu_0^2 \left(\frac{\sigma R \omega}{3}\right)^2 = 2\mu_0 \left(\frac{\sigma R \omega}{3}\right)^2$$

Now we can find the force on the disk:

$$(\vec{F}_z)_{\text{disk}} = \int_{\text{disk}} (\vec{T} \cdot d\vec{a})_z = \int_{\text{disk}} T_{zz} da_z$$

$T_{zz}$  is constant, so it will come out of the integral.

Also, because we are enclosing the northern hemisphere, the outward normal would point in the  $-\hat{z}$  direction which will introduce a minus sign.

$$\begin{aligned} (\vec{F}_z)_{\text{disk}} &= T_{zz} \int_{\text{disk}} da_z = -T_{zz} \pi R^2 \\ &= -2\pi \mu_0 \left(\frac{\sigma R^2 \omega}{3}\right)^2 = \underline{\underline{(\vec{F}_z)_{\text{disk}}}} \end{aligned}$$

Now we need to find the force on the rest of the x-y plane.

Rest of x-y plane

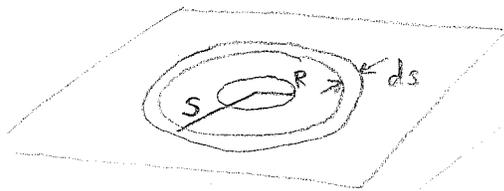
Again  $da_x = da_y = 0 \Rightarrow$  we only need  $T_{zz}$ .

Now we are outside the shell. In the x-y plane, the  $\vec{B}$  field would be

$$\vec{B} = -\frac{\mu_0}{3} (\sigma R \omega) \left(\frac{R}{r}\right)^3 \hat{z} \quad (\theta = 90^\circ, \hat{\theta} = -\hat{z} \text{ in x-y plane})$$

Again we have only z component, but now it decays  $(\propto \frac{1}{r^3})$ .

Everything has  $\phi$  rotational symmetry so we can choose  $da_z$  as rings.



Again a minus sign will be introduced because the normal points in the  $-\hat{z}$  direction. We again have only a  $z$ -component so

$$T_{zz} = \frac{1}{2\mu_0} B^2$$

$$(F_z)_{\text{plane}} = \int_{\text{plane}} (\vec{T} \cdot d\vec{a})_z = \int_{\text{plane}} T_{zz} da_z$$

$$= - \int_R^\infty \frac{1}{2\mu_0} \mu_0^2 \left(\frac{\sigma R\omega}{3}\right)^2 \frac{R^6}{s^6} 2\pi s ds$$

$$= -\pi \mu_0 \left(\frac{\sigma R\omega}{3}\right)^2 R^6 \int_R^\infty \frac{ds}{s^5} = -\pi \mu_0 \left(\frac{\sigma R\omega}{3}\right)^2 R^6 \left[ \frac{1}{4s^4} \right]_R^\infty$$

$$(F_z)_{\text{plane}} = -\frac{\pi \mu_0}{4} \left(\frac{\sigma R^2 \omega}{3}\right)^2$$

$$F_z = -\pi \mu_0 \left(\frac{\sigma R^2 \omega}{3}\right)^2 \left(\frac{9}{4}\right) = \boxed{-\pi \mu_0 \left(\frac{\sigma R^2 \omega}{2}\right)^2} = F_z$$

For  $S_2$  we would have:

$$F_z = \oint_{S_2} (\vec{T} \cdot d\vec{a})_z = \int_{\text{disk of radius } R} (\vec{T} \cdot d\vec{a})_z + \int_{\text{hemisphere of radius } R} (\vec{T} \cdot d\vec{a})_z$$

We have already found the first integral. Looking at the expression for the force using  $S_1$ , we would expect the integral over the hemisphere of radius  $R$  to give the same result as for the rest of the  $x$ - $y$  plane for  $S_1$  because we know  $F_z$  is the same for both. Let's show that it does work out.

Ex. 8.3 continued

Hemi-sphere

$$\begin{aligned} (\vec{T} \cdot d\vec{a})_z &= T_{zx} da_x + T_{zy} da_y + T_{zz} da_z \\ &= \frac{1}{\mu_0} \left[ B_z B_x da_x + B_z B_y da_y + B_z B_z da_z - \frac{1}{2} B^2 da_z \right] \\ &= \frac{1}{\mu_0} \left[ B_z (\vec{B} \cdot d\vec{a}) - \frac{1}{2} B^2 da_z \right] \end{aligned}$$

$$\vec{B} = \frac{\mu_0 \sigma R \omega}{3} [2 \cos \theta \hat{r} + \sin \theta \hat{\theta}] \quad \text{using } r=R$$

$$d\vec{a} = R^2 \sin \theta d\theta d\phi \hat{r} \Rightarrow \vec{B} \cdot d\vec{a} = \frac{2\mu_0 \sigma R^3 \omega}{3} \sin \theta \cos \theta d\theta d\phi$$

$$\begin{aligned} B_z &= \vec{B} \cdot \hat{z} = \frac{\mu_0 \sigma R \omega}{3} [2 \cos \theta \hat{r} \cdot \hat{z} + \sin \theta \hat{\theta} \cdot \hat{z}] \\ &= \frac{\mu_0 \sigma R \omega}{3} [2 \cos^2 \theta - \sin^2 \theta] = \frac{\mu_0 \sigma R \omega}{3} [3 \cos^2 \theta - 1] \end{aligned}$$

$$B^2 = \frac{\mu_0^2 \sigma^2 R^2 \omega^2}{9} [4 \cos^2 \theta + \sin^2 \theta] = \left( \frac{\mu_0 \sigma R \omega}{3} \right)^2 [3 \cos^2 \theta + 1]$$

$$da_z = d\vec{a} \cdot \hat{z} = R^2 \sin \theta \cos \theta d\theta d\phi$$

$$(\vec{T} \cdot d\vec{a})_z = \left( \frac{2}{9} \mu_0 \sigma^2 R^4 \omega^2 \sin \theta \cos \theta [3 \cos^2 \theta - 1] - \frac{1}{2} \frac{\mu_0 \sigma^2 R^4 \omega^2}{9} \sin \theta \cos \theta [3 \cos^2 \theta + 1] \right) d\theta d\phi$$

$$= \frac{\mu_0 (\sigma R^2 \omega)^2}{2} \sin \theta \cos \theta [12 \cos^2 \theta - 4 - 3 \cos^2 \theta - 1] d\theta d\phi$$

$$= \frac{\mu_0 (\sigma R^2 \omega)^2}{2} \sin \theta \cos \theta [9 \cos^2 \theta - 5] d\theta d\phi$$

$$(F_z)_{\text{hemi}} = \mu_0 \pi \left( \frac{\sigma R^2 \omega}{3} \right)^2 \int_0^{\pi/2} \sin \theta [9 \cos^3 \theta - 5 \cos \theta] d\theta$$

$\phi$  integral is trivial (symmetry)

$$= \mu_0 \pi \left( \frac{\sigma R^2 \omega}{3} \right)^2 \left[ \frac{9}{4} \cos^4 \theta + \frac{5}{2} \cos^2 \theta \right]_0^{\pi/2}$$

$$= \mu_0 \pi \left( \frac{\sigma R^2 \omega}{3} \right)^2 \left[ \frac{9}{4} - \frac{5}{2} \right]$$

$$(F_z)_{\text{hemi}} = -\frac{\mu_0 \pi}{4} \left( \frac{\sigma R^2 \omega}{3} \right)^2$$