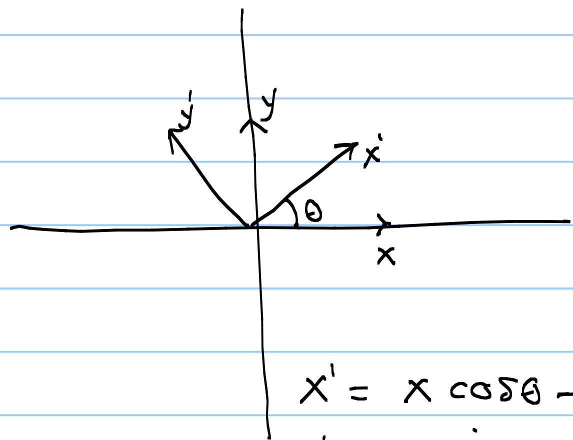
Note Title 10/5/2007

10-5-07 Rotations



$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{Q} \begin{pmatrix} x \\ y \end{pmatrix}$$

Now, Q^{-1} must be a rotation by $-\theta$

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta$$

$$\Rightarrow \quad Q^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Proof: $$QQ^{-1} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

$$= \begin{pmatrix} c^2+s^2 & sc-cs \\ cs-sc & s^2+c^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

similarly for $Q^{-1}Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

examples

45° rotation

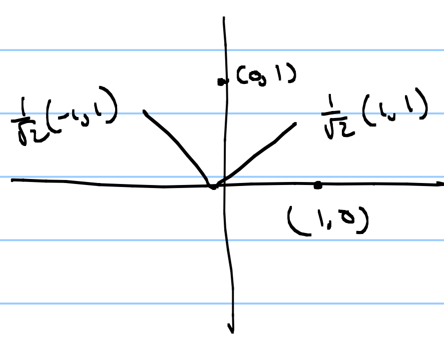
$\cos\theta = \frac{1}{\sqrt{2}} = \sin\theta$

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Apply this to the basis vectors

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

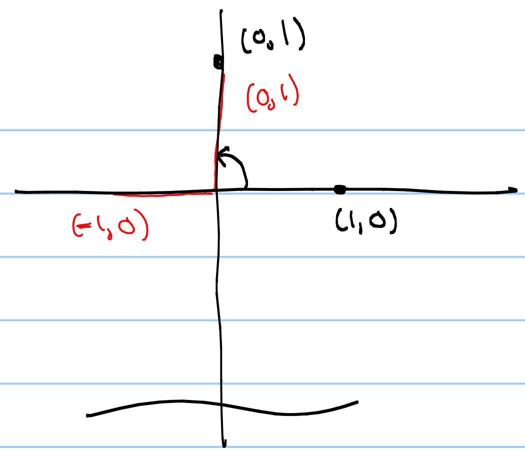
$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



90° rotation

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$



It must be that
2 45° rotation = 1 90°

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} \stackrel{?}{=} \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1-1 & -1-1\\ 1+1 & -1+1\end{pmatrix}$$

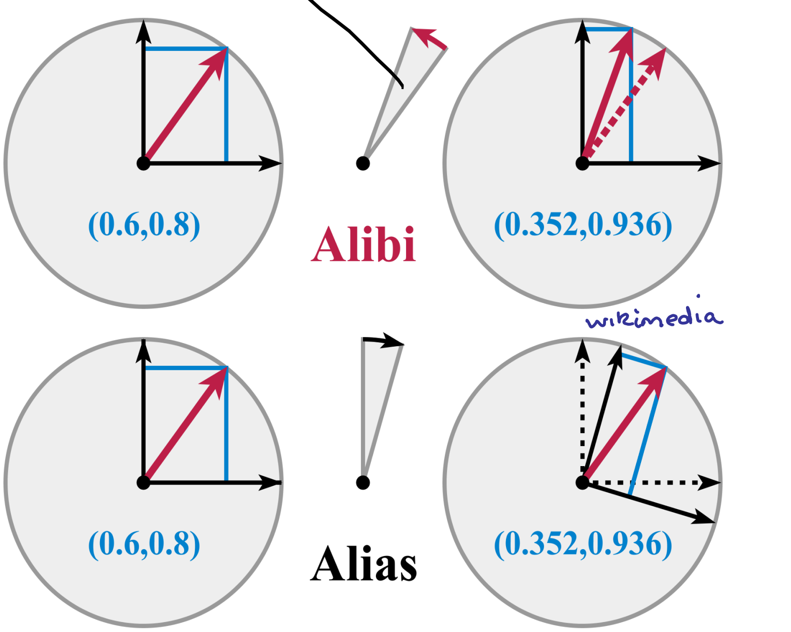
$$= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

Two equivalent ways to think about rotations

Alias — Alibi

2 different names for same thing	same thing is some where else
A vector represented by 2 sets of co-ordinates rotated by θ	A vector rotated by θ in a fixed coord. system

$16.27^\circ \doteq .284$ radians



They work the same way Algebraically

$$\underbrace{\begin{pmatrix} r_x' \\ r_y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

same $\vec{r}$ in primed + unprimed

$\vec{r} \to \vec{R}$ by rotation

$$\begin{pmatrix} R_x \\ R_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

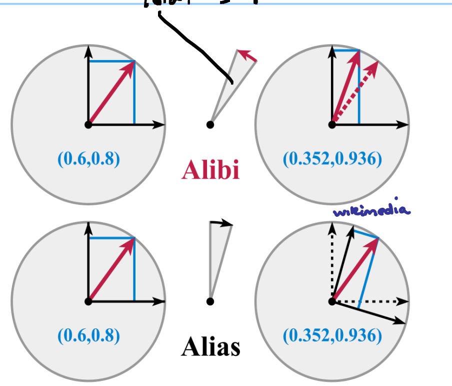
for the example above

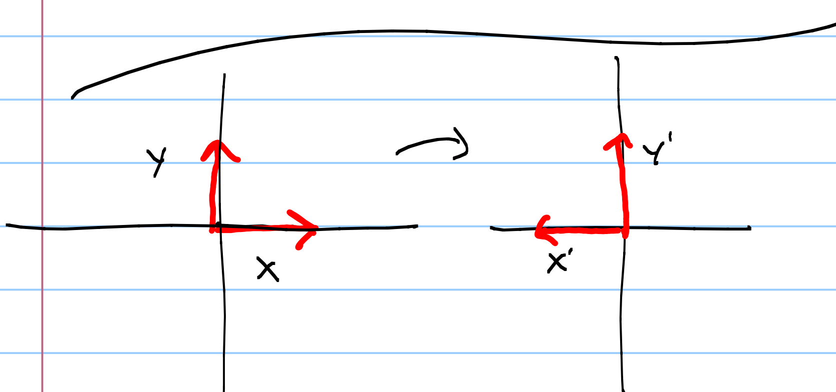
$\theta = 16.27^\circ \approx .284$ radians

$$Q = \begin{pmatrix} .96 & -.28 \\ .28 & .96 \end{pmatrix}$$

$$Q \cdot \begin{pmatrix} .6 \\ .8 \end{pmatrix} = \begin{pmatrix} .352 \\ .936 \end{pmatrix}$$

$16.27^\circ \cong .284$ radians





$$\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \begin{pmatrix}-1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}0\\1\end{pmatrix}$$

What is the transformation matrix?

$$\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \begin{matrix}-1 = a\\0 = c\end{matrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1 & b\\0 & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \begin{matrix}b=0\\d=1\end{matrix}$$

$$\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$$ verify that

$$Q^T Q = Q Q^T = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

is this a rotation?

in original coord. $\hat{x} \times \hat{y} = \hat{z}$
in new coord. $\hat{x} \times \hat{y} = -\hat{z}$

This is a reflection

$$\text{Det}\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix} = -1$$

Rotations satisfy

$$\boxed{\begin{gathered} Q^T Q = Q Q^T = I \\ \text{Det}(Q) = +1 \end{gathered}}$$

what is the general expression for a 2×2 rotation?

Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad M^T M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} a^2 + c^2 & ba + cd \\ ba + cd & b^2 + d^2 \end{pmatrix}$$

notice that $M^T M$ is <u>always symmetric</u>

we want $M^T M = I$

so $\Rightarrow$

$$a^2 + c^2 = 1$$
$$b^2 + d^2 = 1$$
$$ba + cd = 0$$

Let $\vec{r}_1 = \begin{pmatrix} a \\ c \end{pmatrix}$ $\quad \vec{r}_2 = \begin{pmatrix} b \\ d \end{pmatrix}$

then $a^2 + c^2 = 1 \iff \|\vec{r}_1\| = 1$

$b^2 + d^2 = 1 \iff \|\vec{r}_2\| = 1$

$ba + cd = 0 \iff \vec{r}_1^{\,T} \vec{r}_2 = 0$

One last condition

$\det(M) = 1 \qquad ad - cb = 1$

this says that for $\vec{r}_1, \vec{r}_2$ in the x-y plane of $\mathbb{R}^3$ that $\vec{r}_1 \times \vec{r}_2 = \hat{z}$

i.e. $\vec{r}_1$ and $\vec{r}_2$ are oriented to form a RH set of vectors.

infinitesimal rotations

$$\underbrace{Q(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{finite rotation}}$$

small angles $\cos\theta \to 1$ $\sin\theta \to \theta$

$$dQ(\theta) = \begin{pmatrix} 1 & -d\theta \\ d\theta & 1 \end{pmatrix}$$

lets verify that $(dQ)^T (dQ) = I$

$$\begin{pmatrix} 1 & d\theta \\ -d\theta & 1 \end{pmatrix} \begin{pmatrix} 1 & -d\theta \\ d\theta & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1+d\theta^2 & -d\theta+d\theta \\ -d\theta+d\theta & 1+d\theta^2 \end{pmatrix}$$

So to first order in $d\theta$ (ie neglecting $d\theta^2$ compared to $d\theta$)

$$dQ^T dQ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Show that $(dQ)^2 =$

$$\begin{pmatrix} 1 & -2d\theta \\ 2d\theta & 1 \end{pmatrix}$$