MATH 332 - Linear Algebra August 3, 2009

Homework 9, Summer 2009 **Due**: August 5, 2009

Least Squares - Orthogonal Diagonalization - Spectral Decomposition - Singular Value Decomposition

1. Given,

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ -5 & -1 \\ 1 & 2 \end{bmatrix}.$$

Determine an orthonormal basis for the column space of $\mathbf{A}$.

2. Given the linear system of equations,

$$x_1 + x_2 = 2$$

$$x_1 + x_2 = 4$$

(a) Determine the least-squares solution to the linear system.

(b) Determine the least-squares error associated with the linear system.

(c) Graph the linear system, the least-squares solution, and the least-squares error in $\mathbb{R}^2$.

3. Given,

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}.$$

(a) Show that the columns of $\mathbf{A}$ are linearly independent.

(b) Determine the **QR** factorization of $\mathbf{A}$.

(c) Using this factorization calculate the unique least-squares solution $\hat{\mathbf{x}} = \mathbf{R}^{-1}\mathbf{Q}^{\mathrm{T}}\mathbf{b}$.[1]

4. Recall the Pauli Spin Matrix from a previous homework,

$$\sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

(a) Show that σ_y is self-adjoint. [2]

(b) Find the orthogonal diagonalization of σ_y.[3]

(c) Show that $\sigma_y = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^{\mathrm{H}} + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^{\mathrm{H}}$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the normalized eigenvectors from part (b).[4]

5. Given,

$$\mathbf{A} = \begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix}.$$

Find a singular value decomposition of $\mathbf{A}$.

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[1]See theorem 6.5.15 on page 414.

[2]If a matrix is equal to its transpose then we say that the matrix is symmetric. If the matrix has complex numbers then we have a more general definition. For $\mathbf{A} \in \mathbb{C}^{n \times n}$ we say that $\mathbf{A}$ is self-adjoint if $\mathbf{A}^{\mathrm{H}} = \bar{\mathbf{A}}^{\mathrm{T}} = \mathbf{A}$. That is, a matrix is self-adjoint if it is equal to its own complex-conjugate transpose. Self-adjoint matrices are the analouge of symmetric matrices for the complex number field. Also, notice that this definition recovers the definition of symmetric if the matrix has only real entries.

[3]You should find eigenvectors with complex entries. If you use the standard definition of inner-product then you will get zero length, which is sensible since part of the direction of these vectors is into the complex number system. However, this will lead you to a division by zero when trying to normalize the eigenvector. In the case where vectors have complex entries the inner-product is generalised to $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^{\mathrm{H}}\mathbf{y} = \bar{\mathbf{x}}^{\mathrm{T}}\mathbf{y}$. Notice, again that the standard definition of inner-product is recovered when the vectors are real.

[4]This is called the spectral decomposition of a self-adjoint matrix. It is interesting to note that $\mathbf{x}_i^{\mathrm{H}}\mathbf{x}_i = 1$, which implies that the 'matrix' has the same structure as the unitary matrices of chapter 6. Since, $\mathbf{x}_i\mathbf{x}_i^{\mathrm{H}}$ is not the identity matrix we have that $\mathbf{x}_i\mathbf{x}_i^{\mathrm{H}}$ has the same structure as unitary matrices from problem 3 in homework 8. Consequently, we conclude that the self-adjoint matrix has been decomposed into projection matrices, which project an arbitrary vector into eigen-subspaces of the original matrix.