Propagation of E, B fields:

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad \vec{k} \text{ complex}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\partial_t \vec{B} = \frac{i\omega}{c}\vec{B} = i\tilde{k}(\hat{k}\times\vec{E})$$

$$\vec{B} = \frac{\tilde{k}c}{\omega}(\hat{k}\times\vec{E}) = \tilde{n}\,\hat{k}\times\vec{E}$$

$\vec{B} \perp \vec{E} \perp \vec{k}$ still

complex $\tilde{n} \rightarrow$ phase diff btw B, E

Magnitude of B in medium:

$$B_0 = |\vec{B}| = |\tilde{n}|\, E_0 = |\sqrt{\tilde{\varepsilon}\mu}\,|\, E_0$$

generally $B_0 > E_0$ inside medium

if conductivity is <u>low</u> $\quad \frac{4\pi\sigma}{\varepsilon\omega} \ll 1$

$$\tilde{n} = \sqrt{\varepsilon\mu}\left(1 + i\frac{4\pi\sigma}{\varepsilon\omega}\right)^{1/2} \approx \sqrt{\varepsilon\mu}\left(1 + \frac{1}{2}i\frac{4\pi\sigma}{\varepsilon\omega}\right)$$

$\rightarrow$ damped wave $\quad e^{ik_0 n_R z}\, e^{-k_0 n_I z} \qquad$ with $n_I \approx \sqrt{\frac{\mu}{\varepsilon}}\frac{2\pi\sigma}{\omega}$

high conductivity $\quad \tilde{\varepsilon} = \varepsilon + i\frac{4\pi\sigma}{\omega} \sim i\frac{4\pi\sigma}{\omega}$ (extreme limit)

$$\rightarrow \tilde{k} = k_0\sqrt{\tilde{\varepsilon}\mu} = \frac{\omega}{c}\sqrt{\frac{4\pi\sigma\mu}{\omega}}\cdot\sqrt{i}$$

$$\sqrt{i} = e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$$

wave is strongly damped

$$k_I = \frac{1}{c}\sqrt{2\pi\mu\sigma\omega}$$

characteristic damping length $\quad 1/k_I = \frac{c}{\sqrt{2\pi\mu\sigma\omega}} \equiv \delta$

δ = skin depth.

- wave is pushed out of conductor, reflects.
- $|\vec{B}| \gg |\vec{E}|$, $45°$ phase shift.
- surface currents from B field

Next, look at skin effect : very low freq, $\vec{E}$ can penetrate

The skin effect

calculate $\vec{E}$ in a conducting wire for AC current.

start with wave eqn:

$$\nabla^2 \vec{E} - \frac{4\pi\sigma\mu}{c^2}\frac{\partial \vec{E}}{\partial t} = \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

Claim: very good conductor, throw out term on RHS

show — $E \sim e^{-i\omega t}$

$$\rightarrow \nabla^2 E + i\frac{4\pi\sigma\mu\omega}{c^2}\vec{E} = -\frac{\omega^2\varepsilon\mu}{c^2}\vec{E}$$

compare magnitude of the terms

$4\pi\sigma$ vs. $\omega\varepsilon$

if $\omega \ll \frac{4\pi\sigma}{\varepsilon}$ (either low freq, or high σ limit)

we can drop RHS, leaving a diffusion eqn:

$$\nabla^2 \vec{E} - \frac{4\pi\sigma\mu}{c^2}\frac{\partial \vec{E}}{\partial t} = 0$$

looks like Schrödinger eqn, but S.E. has $i\hbar\, \partial\psi/\partial t$

turn this into an eqn. for $\vec{J} = \sigma\vec{E}$,

use $\vec{J} = \vec{J}_0 e^{-i\omega t}$

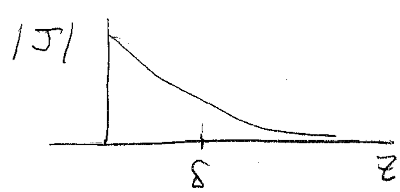
$$\nabla^2 \vec{J}_0 + k^2 \vec{J}_0 = 0 \qquad \text{with} \quad k^2 \equiv i\,\frac{4\pi\sigma\mu\omega}{c^2}$$

since $\sqrt{i} = \frac{1+i}{\sqrt{2}}$ $\quad k = (1+i)\frac{\sqrt{2\pi\sigma\mu\omega}}{c} = \frac{1+i}{\delta}$

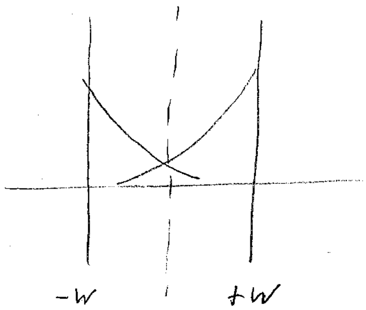
$\delta = c/\sqrt{2\pi\sigma\omega}$ = skin depth.

in slab geometry, solution is

$$\vec{E} = \frac{\vec{J}_0}{\sigma} = \frac{J_0}{\sigma}\hat{x}\, e^{ikz}$$



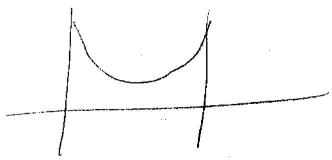
with two surfaces, E on both sides:



$$E(z) = E_0\left[e^{ik(z-w)} + e^{-ik(z-w)}\right]$$

$$= 2E_0 \cosh(k(z-w))$$

so magnitude is



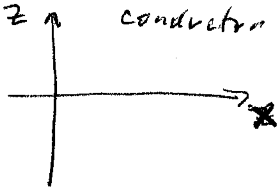
degree of penetration depends on skin depth $\delta \propto 1/\sqrt{\omega}$

for low freq. δ is large, field is uniform throughout.

e^{ikx} dependence

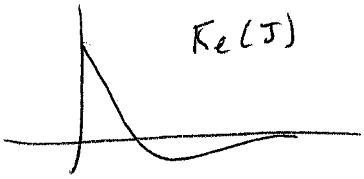
$-k^2 J_0 + k^2 J_0 = 0$

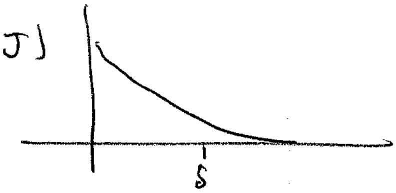
Slab geometry



$\vec{E}, \vec{J}$ in z-direction

$$J_z(x) = J_z(0)\, e^{i(i+1)x/\delta} = J_z(0)\, e^{(i-1)x/\delta}$$





cylindrical geometry (wire)

$$\nabla^2 \vec{J}_0 + k^2 \vec{J}_0 \rightarrow \frac{1}{r}\frac{d}{dr}\left(r \frac{dJ_z}{dr}\right) + k^2 J_z = 0$$

compare to Bessel's eqn.

Abramowitz + Stegun $\qquad z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2) w = 0$

our eqn:

$$\frac{d^2 J_z}{dr^2} + \frac{1}{r}\frac{dJ_z}{dr} + k^2 J_z \rightarrow r^2 \frac{d^2 J_z}{dr^2} + r \frac{dJ_z}{dr} + k^2 r^2 J_z = 0$$

let $\eta = kr$ $\qquad \frac{dJ_z}{dr} = \frac{d\eta}{dr}\frac{dJ_z}{d\eta} = k \frac{dJ_z}{d\eta}$

finally $\rightarrow \eta^2 \frac{d^2 J_z}{d\eta^2} + \eta \frac{dJ_z}{d\eta} + \eta^2 J_z = 0$

$\therefore$ this is Bessel eqn, $\nu = 0$

solutions are

$$J_z(\eta) = c_1 \mathcal{J}_0(\eta) + c_2 \underbrace{N(\eta)}_{\text{diverges at origin}}$$

$$\rightarrow J_z(r,t) = A\, \mathcal{J}_0(kr)\, e^{-i\omega t}$$

(take real pt)

since κ is complex $\Rightarrow$ different functional behavior than "normal J_0"

$\Rightarrow$ ber(x) bei(x)

physics: low ω - current is uniform through wire

$R_{DC} \propto \pi a^2$

high ω - current expelled from interior.

$R_{AC} \propto 2\pi a \cdot \delta$ i.e. outer sheath along circumf.