Mode-locking: homogeneous line Appx F

Just as the beam must reproduce itself on each round trip in a stable cavity, a mode-locked pulse must have the same temporal profile on each round trip.

We'll develop a propagation equation that will wash out the discrete elements of the ~~amplif~~ oscillator.

- assume fluoresc. lifetime $\tau > t_{RT}$
- saturated gain:

$$g = \frac{g_0}{1 + I/I_s} \qquad I_s = \frac{h\nu}{\sigma_{pk}} \cdot \frac{1}{\tau}$$

g_0 = unsaturated gain

at steady-state,

$$g = \sigma_{pk} N_0$$

N_0 = steady-state inversion density

→ single pass gain e^{gl} at line center.

for any ω: $g(\omega) = g / \left[1 + \left(\frac{2(\omega - \omega_0)}{\Delta\omega_0}\right)^2\right]$ lorentzian.

Represent pulse in a reference frame moving w/ pulse.

$$E(t) = A(t) e^{-i\omega_0 t + \phi} \qquad t = 0 \Rightarrow \text{pulse peak.}$$

spectral domain:

$$A(\omega - \omega_0) = \mathcal{F}\{A(t)\} \equiv \int_{-\infty}^{\infty} A(t) e^{i\omega t} dt$$

Just as with propagation through glass, the gain medium acts to modify the pulse in the spectral domain:

one pass: $A_2(\omega) = A_1(\omega) e^{i\frac{\omega}{c}nl} \exp\left[\frac{gl/2}{1 + \frac{2i(\omega - \omega_0)}{\Delta\omega_0}}\right]$

complex lorentzian

When we worked with pulse propagation in materials, we expanded the spectral phase around ω_0.
Here, if $\omega-\omega_0 << \Delta\omega_0$ propagation phase is:

$$i\phi(\omega) = i\frac{\omega}{c}n l + \frac{gl/2}{1 + 2i\frac{(\omega-\omega_0)}{\Delta\omega_0}}$$

$$= i\frac{\omega}{c}n l + \frac{gl}{2}\frac{\left(1 - 2i\frac{(\omega-\omega_0)}{\Delta\omega_0}\right)}{1 + \frac{4(\omega-\omega_0)^2}{\Delta\omega_0^2}}$$

$$= i\left(\frac{\omega}{c}n l + gl\frac{(\omega-\omega_0)}{\Delta\omega_0}\right) + \frac{gl}{2}\left(1 - \frac{4(\omega-\omega_0)^2}{\Delta\omega_0^2}\right)$$

keeping terms to 2nd order in $(\omega-\omega_0)$

imaginary pt → group delay:

$$\tau_g = \frac{\partial\phi}{\partial\omega} = \left(\frac{n}{c} + \underbrace{\frac{g}{\Delta\omega_0}}_{\text{gain}}\right)l$$

we'll drop this since we'll keep reference frame moving with pulse

∴ concentrate on amplitude:

$$A_2(\omega) = A_1(\omega)\exp\left(\frac{gl}{2}\left(1 - \frac{4(\omega-\omega_0)^2}{\Delta\omega_0^2}\right)\right)$$

After 2 passes: $gl/2 \to gl$
expand exp( ) for small gl:

$$A_3(\omega) = A_1(\omega)\left(1 + gl\left(1 - \frac{4(\omega-\omega_0)^2}{\Delta\omega_0^2}\right)\right)$$

Note that the net effect is to narrow the spectrum:
highest gain at $\omega=\omega_0$, lower for $(\omega-\omega_0)^2 > 0$

What is the effect of the amplifier on the pulse in the time domain?

Fourier trick:

$$\mathcal{F}\left\{ \frac{d^n}{dt^n} A(t) \right\} = \left[i(\omega - \omega_0) \right]^n A(\omega - \omega_0)$$

proof:

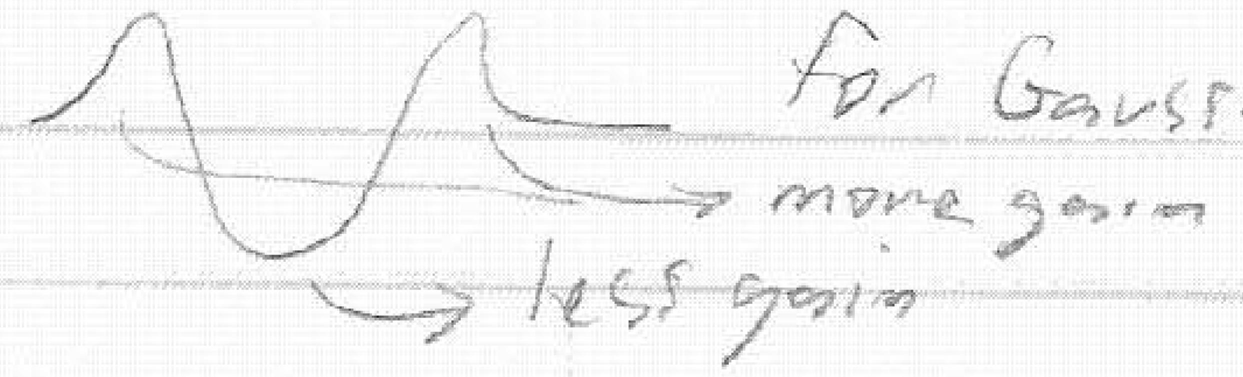
$$\frac{d^n}{dt^n} \left(\int A(\omega - \omega_0)\, e^{-i(\omega-\omega_0)t} \, d(\omega - \omega_0) \right)$$

$$= \mathcal{F}^{-1} \left\{ \left[-i(\omega - \omega_0) \right]^n A(\omega - \omega_0) \right\}$$

$\therefore$ in the time-domain,

$$A_3(t) = \left[1 + g\ell \left(1 + \left(\frac{2}{\Delta\omega_0} \right)^2 \frac{d^2}{dt^2} \right) \right] A_1(t)$$

pulse broadening



for Gaussian.

Lump cavity losses: $e^{-\delta}$ power loss.

fields: $A_4(t) = e^{-\delta/2} A_3(t) \approx (1 - \delta) A_3(t)$

Modulator:

transmission $= e^{-\delta_m (1 - \cos(\omega_m t))}$ double-pass

pulse passes thru when loss is minimized, loss is small

$$\rightarrow 1 - \delta_m (1 - \cos \omega_m t) \approx 1 - \frac{\delta_m}{2} \omega_m^2 t^2$$

Full round trip

$$A_1(t) = \left(1 - \frac{\gamma_m}{2}(\omega_m t)^2\right)(1-\gamma)\left[1 + g\ell\left(1 + \left(\frac{2}{\Delta\omega_0}\right)^2 \frac{d^2}{dt^2}\right)\right] A_1$$

All terms are small,

$$\rightarrow \left[g\ell\left[1 + \left(\frac{2}{\Delta\omega_0}\right)^2 \frac{d^2}{dt^2}\right] - \gamma - \frac{\gamma_m}{2}(\omega_m t)^2 \right] A(t) = 0$$

This is like eqn for SHO:

$$g\ell\left(\frac{2}{\Delta\omega_0}\right)^2 \frac{d^2}{dt^2} A + \left(g\ell - \gamma - \frac{\gamma_m}{2}(\omega_m t)^2\right) A = 0$$

find characteristic timescale: make $A(t) \rightarrow A(x)$

w/ $x = \omega_p t$

$$\rightarrow g\ell\left(\frac{2}{\Delta\omega_0}\right)^2 \omega_p^2 \frac{d^2}{dx^2} A + \left(g\ell - \gamma - \frac{\gamma_m}{2}\frac{\omega_m^2}{\omega_p^2} x^2\right) A = 0$$

multiply all through by $\frac{2}{\gamma_m}\frac{\omega_p^2}{\omega_m^2}$, look at 1st coeff (of d^2/dx^2)

$$\frac{8\, g\ell}{\Delta\omega_0^2 \omega_m^2 \gamma_m} \omega_p^4 \rightarrow 1 \quad \therefore \omega_p = \left(\frac{\gamma_m}{2g\ell}\right)^{\frac{1}{4}} \left(\frac{\omega_m \Delta\omega_0}{2}\right)^{1/2}$$

now eqn reads

$$\frac{d^2}{dx^2} A + \frac{2}{\gamma_m}\frac{\omega_p^2}{\omega_m^2}(g\ell - \gamma) A - x^2 A = 0$$

if $1 - \gamma/g\ell = \frac{4\omega_p^2}{\Delta\omega_0^2}$ then middle coeff $\rightarrow 1$

$$\rightarrow \frac{d^2}{dx^2} A + (1 - x^2) A = 0 \rightarrow A = A_0 e^{-x^2/2}$$

calculate pulse duration, $\Delta\tau_p$ = FWHM intensity

convert $\omega_m = 2\pi\nu_m$ $\quad \Delta\omega_0 = 2\pi\Delta\nu_0$

$$\rightarrow \Delta\tau_p \sim 0.45 \left(\frac{g\ell}{\delta_m}\right)^{1/4} \left(\frac{1}{\nu_m \Delta\nu_0}\right)^{1/2}$$

$\rightarrow \Delta\tau_p$ limited by ν_m

balance of $\frac{\text{saturated gain}}{\text{modulation depth}}$

$\therefore$ can't reach full BW.

estimate $\Delta\tau_p \sim \frac{0.45}{\sqrt{\nu_m \Delta\nu_0}} \rightarrow \omega_p$

then

$$1 - \frac{\epsilon}{g\ell} = \frac{4\omega_p^2}{\Delta\omega_0^2} \rightarrow g\ell$$

$$g = \sigma_{pk} N_0 \rightarrow N_0$$

for Nd:YAG

$\Delta\nu_0 = 120$ GHz

$\nu_m = 76$ MHz.

$\rightarrow \Delta\tau_p = 150$ps

actual 100ps

$\rightarrow g\ell = \delta_m/5$

Passive mode-locking (fast saturable absorber)

same analysis for gain, passive loss

SA transmission $e^{-\gamma_{sa}/2}$

$$\gamma_{sa} = \frac{\gamma'}{1 + I/I_s} \qquad \gamma' = \text{unsaturated loss}$$

normalize A so that $|A|^2 = I$

→ double pass, small γ', $|A|^2/I_s$

$$1 - \gamma' + \gamma' \frac{|A|^2}{I_s}$$

Now get full eqn:

$$\left\{ \underbrace{g'l \left[1 + \left(\frac{2}{\Delta\omega_0}\right)^2 \frac{d^2}{dt^2} \right]}_{\text{saturated gain when SA is saturated.}} - \underbrace{\gamma_c}_{\text{cavity loss w/o SA}} - \gamma' + \gamma' \frac{|A|^2}{I_s} \right\} A(t) = 0$$

this is similar to a soliton eqn.

soln

$$A(t) = A_0 / \cosh(t/\tau_p)$$

$$\tau_p = \left(\frac{2g'l}{\gamma'}\right)^{1/2} \left(\frac{2}{\Delta\omega_0}\right) \left(\frac{I_s}{|A_0|^2}\right)^{1/2}$$

shorter pulse for large modulation depth γ', high $|A_0|^2/I_s$