

Propagation phase

$$\text{general wave } \vec{E}(z, t) = E_0 \hat{x} e^{ik(z - vt)}$$

$k = k_0 n$ in medium

we want to know how wave changes with propagation.

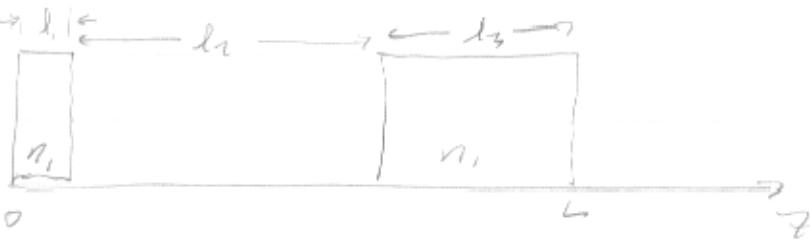
$$\text{at } z=0 \quad \vec{E}(0, t) = E_0 \hat{x} e^{-iwt}$$

$$\text{at } z=l \quad \vec{E}(l, t) = E_0 \hat{x} e^{-iwt} e^{iknl}$$

wave picks up a phase shift of $\phi = knl$
 $n l \equiv \text{"optical path"}$

Note that the wave is continuous (CW): it doesn't matter what time t we pick.

Example:



wave at $z=L = d_1 + d_2 + d_3$

$$\begin{aligned} \vec{E}(L, t) &= E_0 \hat{x} e^{ikn_1 d_1} e^{ikn_2 d_2} e^{ikn_1 d_3} e^{-iwt} \\ &= E_0 \hat{x} e^{ikn_1(n_1 d_1 + d_2 + d_3)} e^{-iwt} \end{aligned}$$

- we're just carrying e^{-iwt} around, often express this
- wave for general $z > L$:

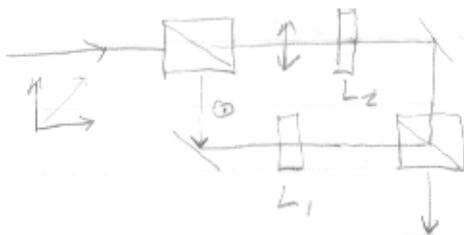
$$\vec{E}(L, t) \cdot e^{ik(z-L)}$$

Review of polarization states:

$$45^\circ \text{ linear } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$L \text{ circular } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} \left(e^{i\pi/4} \right)$$

∴ introducing a $90^\circ (\pi/2)$ relative phase change btwn E_x, E_y
converts linear to circular!



$$E_0 \begin{pmatrix} 1 \\ e^{ik_0 n L_1} \end{pmatrix} + E_0 \begin{pmatrix} 1 \\ e^{ik_0 n (L_2 - L_1)} \end{pmatrix}$$

here phase difference comes
from lengths $\Delta\phi = k_0 n (L_2 - L_1)$

can also use same length, different n !
optimal path (nL) is what matters.

Introduction to birefringence

dielectric constant ϵ is anisotropic

\rightarrow index of refraction $n \equiv \sqrt{\epsilon}$ varies with direction of \vec{E}

isotropic: $\vec{D} = \epsilon \vec{E}$ ϵ constant, scalar

anisotropic: $\vec{D} = \overleftrightarrow{\epsilon} \cdot \vec{E}$ ϵ tensor, 3×3 matrix

- general case:

\vec{D} not parallel to \vec{E}

- coordinates can be chosen to make $\overleftrightarrow{\epsilon}$ diagonal:

$$\rightarrow \vec{D} = \begin{pmatrix} \epsilon_x & & \\ & \epsilon_y & \\ & & \epsilon_z \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

so if \vec{E} is regular along these crystal axes,

E_x "sees" $n_x = \sqrt{\epsilon_x}$,

$E_y \rightarrow n_y$ etc.

uniaxial: $\epsilon_y = \epsilon_z \rightarrow n_o^2$ "ordinary" $n_e = \epsilon_x^{1/2}$ "extraordinary"

Example: $i(k_0 z - \omega t)$

$$\vec{E} = E_0 \begin{pmatrix} a \\ b \end{pmatrix} e^{-i k_0 z - \omega t} \quad \text{in vacuum.}$$

if crystal is aligned with axes along X, Y

pass thru crystal:

$$\vec{E}(l) = E_0 \begin{pmatrix} a e^{ik_0 n_x l} \\ b e^{ik_0 n_y l} \end{pmatrix} e^{-i \text{int. phase}}$$

$z \rightarrow l$

what is polarization state? look at phase difference

$$\begin{pmatrix} a \\ b e^{i k_0 (n_y - n_x) l} \end{pmatrix}$$

Now we have control over polarization state.
look at phase difference

$$\Delta\phi = k_0(n_y - n_x)l$$

full waveplate "λ"
 $\Delta\phi = 2\pi$ (or multiple)
 $e^{i\Delta\phi} = 1 \rightarrow$ no change

half wave plate " $\lambda/2$ "
 $\Delta\phi = \pi$ (or odd multiple)

$$e^{i\Delta\phi} = -1$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} a \\ -b \end{pmatrix}$$

still linear

 polarization is rotated.
by 2θ

special case: $a=b$

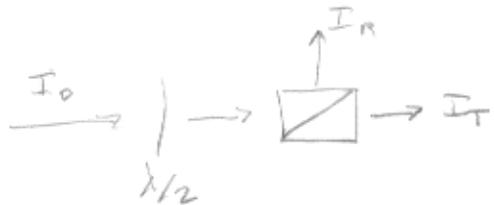
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

rotation by 90°

$$45^\circ \rightarrow -45^\circ$$

Applications:

- variable attenuator:



- Pockels cell: high voltage on crystal \rightarrow waveplate
 \rightarrow fast (ns) optical switch.

quarter wave plate "λ/4"

$\Delta\phi = \pi/2$ or odd multiples

if linear and 45° input

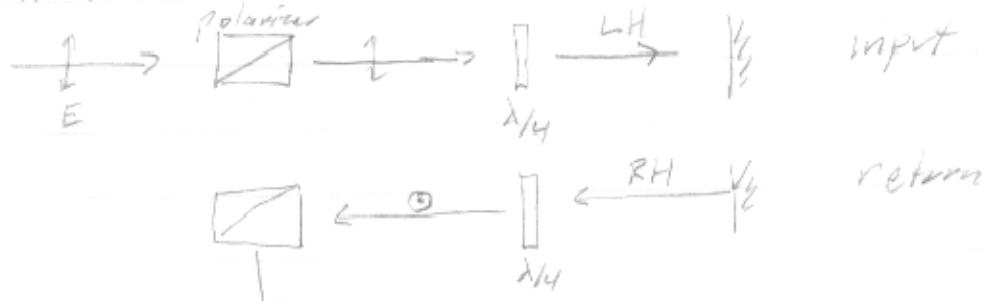
$$\begin{pmatrix} 1 \\ i \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ e^{i\pi/2} \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{circular (Left)}$$

or, rotate $\lambda/4$ plate by 90°

then $n_y - n_x \rightarrow n_x - n_y$

$$\begin{pmatrix} 1 \\ i \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ e^{-i\pi/2} \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{RH circ.}$$

Application: isolator



E_x gets phase $e^{ik_{x,y}z} \cdot e^{ik_{x,y}z} \rightarrow \phi_x = 2k_{x,y}z$
 E_y $\rightarrow \phi_y = 2k_{x,y}z$

i. if $\Delta\phi = \pi/2$ on one pass

$$\Delta\phi = \pi \text{ on 2}$$

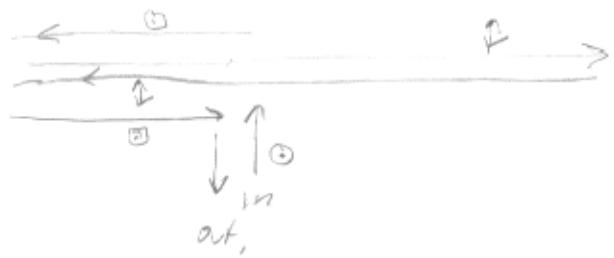
- prevent feedback into laser. (or damage)
- CP read out



"regenerative" amplifier for short pulses ($< 1 \text{ ns}$)



PC off \rightarrow 4 pulses



turn PC on to $\lambda/4$ voltage after 1st double pass
pulse is trapped inside

turn PC off after # round trips (e.g. 2nd)
dump pulse out.

pulse selector

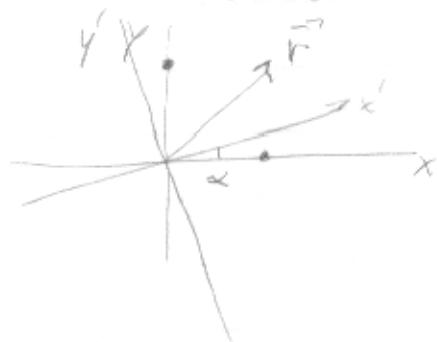


PC off : no light out

PC on to $\lambda/2$ \rightarrow pass pulse

PC off : reject pulses

Coordinate transformations with matrices



$$\vec{r} = (x, y)$$

$$\vec{r}' = (x', y')$$

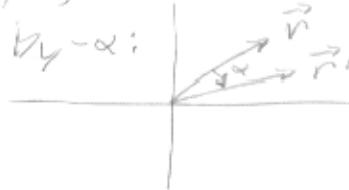
$$x' = x \cos \alpha + y \sin \alpha$$

$$y' = -x \sin \alpha + y \cos \alpha$$

write as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \vec{r}' = R_\alpha \cdot \vec{r}$$

here we are rotating the coordinate system, not the vector.
we can also imagine keeping the coordinate system fixed,
and rotating the vector by $-\alpha$:



Properties of rotation matrices:

inverse: $R_\phi^{-1} = R_{-\phi}$

$$R_\phi R_{-\phi} = I$$

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} c^2+s^2 & 0 \\ 0 & c^2+s^2 \end{pmatrix} = I$$

eigenvalues:

$$\text{eqn } M \vec{r}_\lambda = \lambda \vec{r}_\lambda \quad \lambda = \text{eigenvalue}$$

\vec{r}_λ = eigenvector

find λ :

$$R_\phi \vec{r}_\lambda - I \vec{r}_\lambda = 0 \rightarrow \begin{bmatrix} \cos \phi - \lambda & \sin \phi \\ -\sin \phi & \cos \phi - \lambda \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} =$$

$$\therefore \det(R_\phi - I\lambda) = 0$$

$$(\cos \phi - \lambda)^2 + \sin^2 \phi = 0$$

$$\cos^2 \phi - 2\lambda \cos \phi + \lambda^2 + \sin^2 \phi = 0$$

→ quadratic eqn for 2 roots:

$$\lambda^2 - 2\cos \phi \lambda + 1 = 0$$

$$\lambda = \frac{-(-2\cos \phi) \pm \sqrt{4\cos^2 \phi - 4}}{2} = \cos \phi \pm \sqrt{-\sin^2 \phi}$$

$$= e^{\pm i\phi}$$

eigenvectors: $(\cos \phi - e^{\pm i\phi}) a + (\sin \phi) b = 0$

$$\rightarrow b = \frac{\cos \phi - e^{\pm i\phi}}{\sin \phi} a$$

$$= \frac{\cos \phi - \cos \phi \mp i \sin \phi}{\sin \phi} = \mp i a$$

normalize $\vec{r}_\lambda^\dagger \vec{r}_\lambda = 1$

$$|c|^2 (1 \mp i) \begin{pmatrix} 1 \\ \mp i \end{pmatrix} = 2$$

eigenvectors are: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{for } \lambda = e^{-i\phi}$

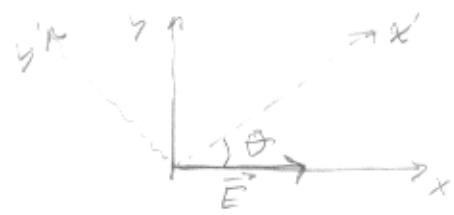
$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \lambda = e^{+i\phi}$$

Note that these correspond to RH, LM circ. polariz!
 These states are the fundamental polarization states
 of photons $\pm h$ spin angular momentum.

Arbitrary angle w.r.t. crystal axes:

$$E_x = E_0$$

$$E_y = 0$$



crystal axes are rotated by θ

$$E_x' = E_0 \cos \theta$$

$$E_y' = -E_0 \sin \theta$$

$$\rightarrow E_0 \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix}$$

now can apply crystal phase:

$$E_{\text{out}} = E_0 \begin{pmatrix} \cos \theta e^{i\phi_x} \\ -\sin \theta e^{i\phi_y} \end{pmatrix}$$

Alternate method: use rotation matrix to change basis

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix}$$

Waveplate of arbitrary orientation

express input wave in terms of crystal axes,

$$\hat{E}_{in} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \hat{E}'_{in} = \hat{R}_\phi \cdot \hat{E}_{in}$$
$$= \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin \phi \\ \cos \phi \end{pmatrix}$$

$$\hat{E}'_{out} = \hat{W} \cdot \hat{E}'_{in}$$
$$= \begin{pmatrix} e^{ik_0 n_0 l} & 0 \\ 0 & e^{-ik_0 n_0 l} \end{pmatrix} \begin{pmatrix} \sin \phi \\ \cos \phi \end{pmatrix}$$
$$= e^{ik_0 n_0 l} \begin{pmatrix} \sin \phi \\ e^{-ik_0 (n_0 - n_1) l} \cos \phi \end{pmatrix}$$

If we want, we can express this in the original basis

$$\hat{E}_{out} = \hat{R}_{-\phi} \cdot \hat{E}'_{out}$$

$$= \hat{R}_{-\phi} \underbrace{\hat{W} \hat{R}_\phi}_{\hat{R}^{-1} \hat{W} \hat{R}} \hat{E}'_{in}$$

$\hat{R}^{-1} \hat{W} \hat{R}$ = \hat{W}' rotated matrix

Jones matrices for waveplates

in crystal axis coords, matrix is diagonal:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \rightarrow \begin{pmatrix} e^{ik\alpha n_1} & 0 \\ 0 & e^{ik\alpha n_2} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

can pull out constant phase factor: (and drop it)

$$\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & e^{i\Delta\phi} \end{pmatrix}$$

$$\text{for } \lambda/4: \Delta\phi = \pi/2 \rightarrow M_{\lambda/4} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$M_{\lambda/2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for arbitrary notation,

$$R_{-\phi} M R_\phi$$

$$\text{ex. rotate by } 45^\circ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ x & x \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1+x & -1+x \\ -1+x & 1+x \end{pmatrix}$$

$$\text{rotated } \lambda/2: x = -1 \rightarrow \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\text{test: H-pol input } \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad 90^\circ \text{ not } \checkmark$$

$$\text{rotated } \lambda/4: x = i \rightarrow \frac{1}{2} \begin{pmatrix} 1+i & -1+i \\ -1+i & 1+i \end{pmatrix}$$

$$\text{test: H-pol input } \rightarrow \frac{1}{2} \begin{pmatrix} 1+i \\ -1+i \end{pmatrix}$$

$$\text{convert to polar: } \frac{1}{2}(1+i) = \frac{1}{\sqrt{2}} e^{i\pi/4}$$

$$\text{factor out common phase: } \frac{1}{2}(-1+i) = \frac{1}{\sqrt{2}} e^{3i\pi/4}$$

$$\rightarrow \frac{1}{\sqrt{2}} e^{i\pi/4} \begin{pmatrix} 1 \\ e^{i\pi/4} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \checkmark$$