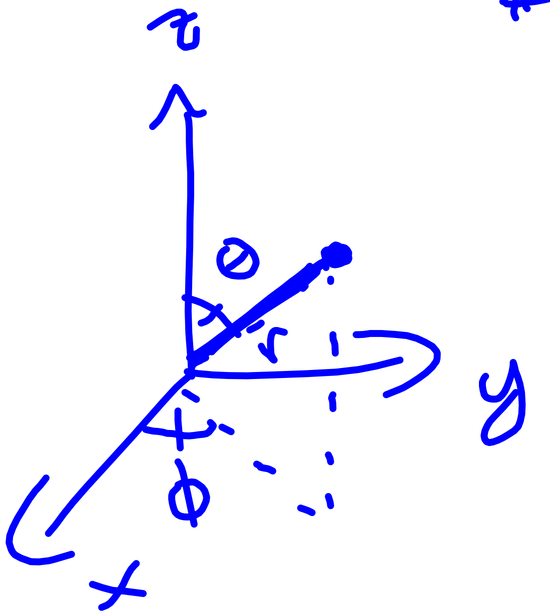
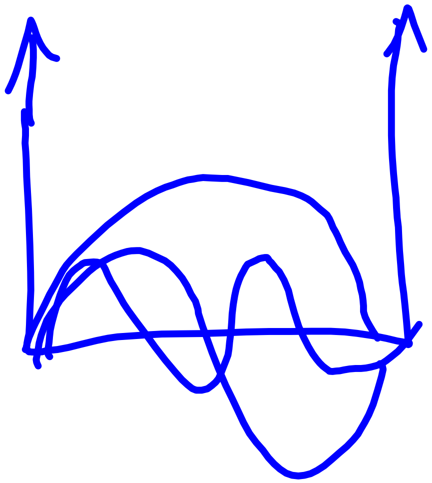
Reading: Today: 11.1

Tomorrow: 11.2



z
θ
r
φ
y
x



Spherical Coordinates

A function that's very common to use in sherical problems are spherical harmonics.

$Y_{\ell m}(\theta,\phi)$ is complete in θ,ϕ.

That means that any function can be written as

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\phi)\, c_{\ell m}$$

Some properties are $-\ell \le m \le \ell$

To find $c_{\ell m}$ you multiply both sides by $Y^*_{\ell' m'}(\theta,\phi)$, and then integrate over your space

$$\int f(\theta,\phi)\, Y^*_{\ell' m'}(\theta,\phi)\, d\Omega = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} c_{\ell m} \underbrace{\int Y^*_{\ell' m'} Y_{\ell m}\, d\Omega}_{\delta_{\ell\ell'}\delta_{mm'}}$$

$\sin\theta\, d\theta\, d\phi$

$$c_{\ell' m'} = \int f(\theta,\phi)\, Y^*_{\ell' m'}(\theta,\phi)\, d\Omega$$

$Y_{\ell m}(\theta,\phi) = |Y_{\ell m}\rangle$ $\qquad$ $\langle Y_{\ell' m'}|Y_{\ell m}\rangle = \delta_{\ell'\ell}\,\delta_{m'm}$

orthogonality cond.

One interesting way of writing the completeness condition is

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \phi')\, Y_{\ell m}(\theta, \phi) = \delta(\cos(\theta) - \cos(\theta'))\, \delta(\phi - \phi')$$

$$d\Omega = \sin\theta\, d\theta\, d\phi$$
$$= -d(\cos\theta)\, d\phi$$

$Y_{00} = \frac{1}{\sqrt{4\pi}}$ ← correspond to s states in H-atom

$\ell = 1$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}$$
$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$
$$Y_{1-1} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}$$

$\ell = 2$

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta\, e^{2i\phi}$$
$$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta\, e^{i\phi}$$
$$Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Radiation Problems

There are these things that are called vector spherical harmonics.

$$\vec{Y}_{\ell m} = Y_{\ell m}\hat{r}$$

$$\vec{\Psi}_{\ell m} = r\vec{\nabla}Y_{\ell m}$$

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}$$

$\vec{\nabla}Y_{\ell m}$ is tangent to sphere

$$\vec{\Phi}_{\ell m} = \vec{r}\times\vec{\nabla}Y_{\ell m}$$

$$\vec{Y}_{\ell m}\cdot\vec{\Psi}_{\ell' m'} = \vec{Y}_{\ell m}\cdot\vec{\Phi}_{\ell' m'} = \vec{\Psi}_{\ell' m'}\cdot\vec{\Phi}_{\ell' m'} = \emptyset$$

$$\int \vec{Y}_{\ell m}\cdot\vec{Y}^{*}_{\ell' m'}\,d\Omega = \delta_{\ell\ell'}\delta_{mm'}$$

$$\int \vec{\Phi}_{\ell m}\cdot\vec{\Phi}^{*}_{\ell' m'}\,d\Omega = \ell(\ell+1)\delta_{\ell\ell'}\delta_{mm'}$$

$$\int \vec{\Psi}_{\ell m}\cdot\vec{\Psi}^{*}_{\ell' m'}\,d\Omega = \ell(\ell+1)\delta_{\ell\ell'}\delta_{mm'}$$

$$\vec{\Phi}_{00} = \vec{\Psi}_{00} = \emptyset$$

Title: Jul 27 - 6:10 PM (5 of 8)

What happens when you do something like $\vec{\nabla}\cdot(\ )$
$\vec{\nabla}\times(\ )$

$\vec{\nabla}(\phi)$: $\phi(r,\theta,\phi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\phi_{\ell m}(r)Y_{\ell m}(\theta,\phi)$

any scalar function

$$\vec{\nabla}(\phi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{d\phi_{\ell m}}{dr}\vec{Y}_{\ell m}+\frac{\phi_{\ell m}}{r}\vec{\Psi}_{\ell m}\right)$$

any vector $\vec{A}$

$$\underset{\text{sums}}{\vec{A}=}\ A^{Y}_{\ell m}(r)\vec{Y}_{\ell m}+A^{\Psi}_{\ell m}(r)\vec{\Psi}_{\ell m}+A^{\Phi}_{\ell m}(r)\vec{\Phi}_{\ell m}$$

$$\vec{\nabla}\cdot\left(A^{Y}_{\ell m}(r)\vec{Y}_{\ell m}\right)=\left(\frac{dA^{Y}_{\ell m}}{dr}+\frac{2}{r}A^{Y}_{\ell m}\right)Y_{\ell m}$$

$$\vec{\nabla}\cdot\left(A^{\Psi}_{\ell m}(r)\vec{\Psi}_{\ell m}\right)=-\frac{\ell(\ell+1)}{r}A^{\Psi}_{\ell m}(r)Y_{\ell m}$$

$$\vec{\nabla}\cdot\left(A^{\Phi}_{\ell m}(r)\vec{\Phi}_{\ell m}\right)=\emptyset$$

$$\vec{\nabla}\times\left(A^{Y}_{\ell m}(r)\vec{Y}_{\ell m}\right)=-\frac{1}{r}A^{Y}_{\ell m}(r)\vec{\Phi}_{\ell m}$$

$$\vec{\nabla}\times\left(A^{\Psi}_{\ell m}(r)\vec{\Psi}_{\ell m}\right)=\left(\frac{dA^{\Psi}_{\ell m}}{dr}+\frac{1}{r}A^{\Psi}_{\ell m}\right)\vec{\Phi}_{\ell m}$$

$$\vec{\nabla}\times\left(A^{\Phi}_{\ell m}(r)\vec{\Phi}_{\ell m}\right)=-\frac{\ell(\ell+1)}{r}A^{\Phi}_{\ell m}\vec{Y}_{\ell m}-\left(\frac{dA^{\Phi}_{\ell m}}{dr}+\frac{1}{r}A^{\Phi}_{\ell m}\right)\vec{\Psi}_{\ell m}$$

Rewrite $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$

Magnetic Multipole expansion.

$$\vec{J} = J^{\Phi}_{lm}(r,t)\,\vec{\Phi}_{lm} \qquad \vec{\nabla}\cdot\vec{J} = \phi \quad \{\text{no } \rho\}$$

$$\vec{E} = E^{\Phi}_{lm}(r,t)\,\vec{\Phi}_{lm} \qquad \vec{\nabla}\cdot\vec{E} = \phi$$

For harmonic dependence, everything goes like $e^{+i\omega t}$.

$$\vec{\nabla}\times\vec{E} = -i\omega\vec{B}$$

$$\rightarrow \begin{cases} \dfrac{l(l+1)}{r} E_{lm}(r) = i\omega B^{r}_{lm} \\ \dfrac{dE_{lm}}{dr} + \dfrac{E_{lm}}{r} = i\omega B^{\phi}_{lm} \end{cases}$$

$$\vec{\nabla}\cdot\vec{B} = \phi \rightarrow \frac{dB^{r}_{lm}}{dr} + \frac{2}{r}B^{r}_{lm} - \frac{l(l+1)}{r}B^{\phi}_{lm} = \phi$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0 i\omega\vec{E}$$

$$\rightarrow -\frac{B^{r}_{lm}}{r} + \frac{dB^{r}_{lm}}{dr} + \frac{B^{r}_{lm}}{r} = \mu_0 J^{\Phi}_{lm} + i\omega\mu_0\epsilon_0 E^{\Phi}_{lm}$$

Green's Functions

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r', t_r)}{\mathscr{R}}\, d\tau'$$

Green's function is $\frac{1}{4\pi}\frac{1}{\mathscr{R}}$

Diff. eqn. $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t}\right) V = -\rho/\epsilon_0$

To solve for a Green's function,
solve $\underbrace{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t}\right)}\, V = -\delta^3(\vec{r} - \vec{r}\,')\,\delta(t - t')$

homogeneous eqn. is this $= 0$.

If $\rho(r, t_r) = \rho(r)\, e^{-i\omega t_r}$

$$t_r = t - \frac{\mathscr{R}}{c}$$

$$= \rho(r)\, e^{-i\omega t}\, e^{i\,\cancel{\omega/c}\,k\,\mathscr{R}}$$

$$V(r, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')\, e^{-i\omega t}\, e^{ik\mathscr{R}}}{\mathscr{R}}\, d\tau'$$

$$V(r, t) = +e^{-i\omega t}\int \frac{\rho(r')}{\epsilon_0}\,\frac{e^{ik\mathscr{R}}}{4\pi\mathscr{R}}\, d\tau'$$

Green's function for harmonic distributions is $\frac{e^{ik\mathscr{R}}}{4\pi\mathscr{R}}$