E. Kreyszig, Advanced Engineering Mathematics, 9^{th} **ed.**	Section 12.2, pgs. 538-540
Lecture: Vibrating Strings and the Wave Equation	Module: 15
Suggested Problem Set: { N/A }	Last Compiled : April 25, 2010

E. Kreyszig, Advanced Engineering Mathematics, 9^{th} **ed.**	Section 12.3, pgs. 540-548
Lecture: Separating Variables	Module: 15
Suggested Problem Set: { 2, 9 }	Last Compiled : April 25, 2010

E. Kreyszig, Advanced Engineering Mathematics, 9^{th} **ed.**	Section 12.4, pgs. 548-552
Lecture: d'Alembert Solution and Characteristics	Module: 15
Suggested Problem Set: { 12, 17 }	Last Compiled : April 25, 2010

E. Kreyszig, Advanced Engineering Mathematics, 9^{th} **ed.**	Section 12.8, pgs. 571-579
Lecture: Vibrations of a Rectangular Membrane	Module: 15
Suggested Problem Set: { 17, 20 }	Last Compiled : April 25, 2010

E. Kreyszig, Advanced Engineering Mathematics, 9^{th} **ed.**	Section 12.9, pgs. 579-586
Lecture: Vibrations of a Circular Membrane	Module: 15
Suggested Problem Set: { 10, 11 }	Last Compiled : April 25, 2010

Beginning Quote of Lecture 15

So now, less than five years later, you can go up on a steep hill in Las Vegas and look west, and with the right kind of eyes you can almost see the high water mark – that place where the wave finally broke and rolled back.

Hunter S. Thompson : Fear and Loathing in Las Vegas (1972)

1. Introduction

We finish our study of PDE with the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \triangle u + F(x,y,z,t), \quad c^2 = \frac{T}{\rho} \in \mathbb{R}^+, \tag{1}$$

where T and ρ define the tension and density respectively. The unknown function u represents a displacement from equilibrium position and the function F is assumed to be an external force applied to the system defined at all space-time points in the PDEs domain. This equation is derived in the text within the context of vibrations of ideally elastic material but can more generally be used to study ideal wave motion. [1] [2] Formally, this equation differs from the heat equation by a second derivative in time as opposed to a first derivative. We will see in the following that being second-order in time will allow for time-oscillations and make these solutions quite different in character that solutions to the heat equation. [3]

[1]To see a derivation consult 12.2 and 12.7 of the EK.

[2]This equation, in fact, manifests more generally in the study of Maxwell's equations from electromagnetic theory as well Einstein's equation on a vacuum background.

[3]This is similar to what you found in ODEs. You might remember that solutions to $y' = ay$ were either exponentially growing or decaying in time. It wasn't until you introduced a second derivative in time that oscillatory solutions were found. Moreover, this wouldn't have even been noticed if wasn't for Euler's identity.

2. Fourier Series Solutions

We begin with the study of the homogeneous wave equation on a finite spatial domain and no external forcing. Or mathematically,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{2}$$

where $x \in (0, L)$ and $t \in (0, \infty)$. This equation models a so-called standing wave from zero to L or maybe more physically the vibrations of an elastic one-dimensional string.[4] Just like the heat equation the interface between this string and its environment must be given and so we establish the typical boundary conditions,

$$u(0, t) = 0, \quad u(L, t) = 0, \tag{3}$$

which implies that the displacement from rest-position of the string at the endpoints is zero for all time. [5] So, we consider an ideally elastic 1D string with fixed endpoints, similar to maybe a guitar string, and whose vibrations are initiated by,

$$u(x, 0) = f(x), \tag{4}$$

$$u_t(x, 0) = g(x), \tag{5}$$

where f is the initial deformation of the string and g is the initial velocity of the string. [6] The equations (2)-(5) constitutes a well-posed problem with (2)-(3) being the boundary value problem and (2)-(5) being the initial value problem. To solve this problem we proceed in the same fashion as we did with the heat equation and note the only changes will be to the order of the ODE on time. These changes to the previous arguments give:

- Separation of Variables : Assume that $u(x, t) = F(x)G(t)$ and using this reduce (2) to two ODEs one on time and one on space.

$$G''(t) + \lambda c^2 G(t) = 0, \tag{6}$$

$$F''(x) + \lambda F(x) = 0 \tag{7}$$

  The spatial equation is known as a boundary value problem (BVP) and is new to most of us. There will be infinitely many solutions to this problem and these solutions are nothing more than modes of a Fourier series expansion of u.

- Solve ODEs : If we solve the time and space ODE's we find that,

$$G_n(t) = B_n \cos(\sqrt{\lambda_n} ct) + B_n^* \sin(\sqrt{\lambda_n} ct), \text{ where } B_n, B_n^* \in \mathbb{R}, \lambda_n = \frac{n^2 \pi^2}{L^2}, n = 1, 2, 3, \ldots, \tag{8}$$

$$F_n(x) = \sin(\sqrt{\lambda_n} x), \quad \sqrt{\lambda_n} = \frac{n\pi}{L}, n = 1, 2, 3, \ldots \tag{9}$$

- Form the solution to the IVP via superposition to get,

$$u(x, t) = \sum_{n=1}^{\infty} \left[B_n \cos(\sqrt{\lambda_n} ct) + B_n^* \sin(\sqrt{\lambda_n} ct) \right] \sin(\sqrt{\lambda_n} x), \tag{10}$$

  where $u(x, 0) = f(x)$ implies that $B_n = \frac{2}{L} \int_0^L f(x) \sin(\sqrt{\lambda_n} x) dx$ and $u_t(x, 0) = g(x)$ implies that $B_n^* = \frac{2}{c\sqrt{\lambda_n} L} \int_0^L f(x) \sin(\sqrt{\lambda_n} x) dx$.

The take home message is that, due to the second derivative in time the dynamics of the Fourier modes have changed from deca
So, the Fourier series on space takes care of whatever shape we might like to write down and this shape

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[4]For those interested in musical instruments this could be the vibrations of a guitar string or the air-column of a woodwind instrument.

[5]In the case of $u_x(0, t) = u_x(L, t) = 0$ we imply that the string must be flat at the endpoints for all time but may undergo displacements. In the case that $u_x(0, t) \propto u(0, t)$ we specify that the slope at the endpoint is proportional to the displacement from equilibrium. Again, like the heat equation, this would lead to boundary value problems that cannot be solved by hand.

[6]It is very common to take $g(x) = 0$. That is, it is very common to initiate vibrations by only deforming the elastic medium.

is then given suitable time dynamics from the time-derivatives in the PDE. If we upgrade this to multiple dimensions then the procedures are the same. That is, given,

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \quad x\in(0,L_1),\ y\in(0,L_2),\ t\in(0,\infty) \tag{11}$$

$$u(0,y,t)=u(L_1,y,t)= \qquad u(x,0,t)=u(x,L_2,t)=0, \tag{12}$$

$$u(x,y,0)=f(x,y), \qquad u_t(x,y,0)=g(x,y), \tag{13}$$

we find a solution expressed by a double Fourier series one for the x-dimension and one for the y-dimension and that the frequency of the time-dynamics will depend on both modes. That is,

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[C_{nm}\cos(\sqrt{\lambda_{nm}}ct)+C^*_{nm}\sin(\sqrt{\lambda_{nm}}ct)\right]\sin(\sqrt{k_n}x)\sin(\sqrt{p_m}x), \tag{14}$$

where $\sqrt{k_n}=n\pi/L_1$, $\sqrt{p_m}=m\pi/L_2$ and $\sqrt{\lambda_{nm}}=\sqrt{k_n+p_m}$. This solution models the vibrations of a thin rectangular membrane with fixed edges. [7] What is more interesting is what happens to the solutions of the PDE when you move to different geometries.

3. Non-Cartesian Geometries

Suppose you want to discuss the vibrations of a thin circular membrane of radius R with fixed edges. In this case the boundary condition would be,

$$u=0 \text{ for all } x,y \text{ such that } x^2+y^2=R^2. \tag{15}$$

These boundary conditions are awful and one should always choose a coordinate system that makes the boundary conditions as easy to express as possible. In this case we choose polar coordinates,

$$x=r\cos(\theta),\quad y=r\sin(\theta), \tag{16}$$

$$r=\sqrt{x^2+y^2},\quad \tan(\theta)=\frac{y}{x}, \tag{17}$$

and our solution would then be $u(r,\theta,t)$ instead of $u(x,y,t)$. The boundary condition, in this case, is simply $u(R,\theta,t)=0$, but we now have introduced a problem. Our PDE (11) is still in Cartesian coordinates but our boundary condition is not. To fix this problem we must ask what u_{xx} and u_{yy} are in the polar system, which is a problem resolved by the multivariate chain rule. For the second derivative of u with respect to x we find that,

$$u_{xx}=(u_x(r,\theta,t))_x \tag{18}$$

$$=(u_r r_x+u_\theta\theta_x)_x \tag{19}$$

$$=u_{rx}r_x+u_r r_{xx}+u_{\theta x}\theta_x+u_\theta\theta_{xx} \tag{20}$$

$$=(u_{rr}r_x+u_{r\theta}\theta_x)\,r_x+u_r r_{xx}+(u_{\theta r}r_x+u_{\theta\theta}\theta_x)\,\theta_x+u_\theta\theta_{xx} \tag{21}$$

$$=u_{rr}r_x^2+u_r r_{xx}+2u_{r\theta}\theta_x r_x+u_{\theta\theta}\theta_x^2+u_\theta\theta_{xx} \tag{22}$$

A similar result holds for u_{yy} and from this we conclude that,[8]

$$u_{xx}+u_{yy}=u_{rr}(r_x^2+r_y^2)+u_r(r_{xx}+r_{yy})+u_{r\theta}(\theta_x r_x+\theta_y r_y)+u_{\theta\theta}(\theta_x^2+\theta_y^2)+u_\theta(\theta_{xx}+\theta_{yy}), \tag{23}$$

where,

$$r_x=\frac{x}{r},\quad r_y=\frac{y}{r},\quad r_{xx}=\frac{1}{r}-\frac{x^2}{r^3},\quad r_{yy}=\frac{1}{r}-\frac{y^2}{r^3} \tag{24}$$

$$\theta_x=-\frac{y}{r^2},\quad \theta_y=\frac{x}{r^2},\quad \theta_{xx}=\frac{2xy}{r^4},\quad \theta_{yy}=-\theta_{xx}. \tag{25}$$

Hence the two-dimensional Laplacian in polar coordinates is,

$$u_{xx}+u_{yy}=u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}, \tag{26}$$

[7] See section 12.7 for the details.

[8] To do this we tacitly assume that u is continuous in the r and θ variables so that $u_{r\theta}=u_{\theta r}$.

and our problem statement is,

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right), \quad r \in (0,R),\ \theta \in (0,2\pi],\ t \in (0,\infty) \tag{27}$$

$$u(R,\theta,t) = 0, \tag{28}$$

$$u(r,\theta,0) = f(r,\theta),\ u_t(r,\theta,0) = g(r,\theta). \tag{29}$$

From here there is still a lot of work that needs to be done. The good thing is that it has been done and recorded for the ages. [9] What is important to note is that the spatial ODE that results from separation of variables looks like,

$$r^2W'' + rW' + (\lambda r^2 - n^2)W = 0, \quad n \in \mathbb{N}. \tag{30}$$

This equation is called Bessel's equation of order n and the solution to this problem is given by,[10]

$$W = J_n(\sqrt{\lambda}r) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m\left(\sqrt{\lambda}r\right)^{2m}}{2^{2m+n}m!(n+m)!}, \tag{31}$$

which is called Bessel's function of the first kind of order n. This function is an oscillatory function but its most important feature is that it oscillates without periodicity.[11] Thus the distance between its roots/nodes is not fixed, which is unlike the sine/cosine functions found when we separated variables in Cartesian coordinates. For this reason one can conclude that the vibrational modes of a circle are more complex than that of a rectangle.

4. Characterizations of Solutions

Lastly, we seek to characterize solutions to the wave equation both on finite domains and infinite domains. To this end we present three important properties of solutions to the wave equation:

(1) **Solutions to the wave equation obey conservation of energy.** To see this we note that the energy associated with the vibrations of an ideally elastic medium can be written as,[12]

$$E = \frac{1}{2}\int_{\mathbb{R}^n}(u_t^2 + c^2|\nabla u|^2)dx. \tag{32}$$

If we differentiate E with respect to time and the wave equation conserves energy then we should get zero. Doing so we find,

$$\frac{dE}{dt} = \frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^n}(u_t^2 + c^2|\nabla u|^2)dx \tag{33}$$

$$= \int_{\mathbb{R}^n}(u_t u_{tt} + c^2\nabla u \cdot \nabla u_t)dx \tag{34}$$

$$= \text{integration by parts and arguments about boundary conditions} \tag{35}$$

$$= \int_{\mathbb{R}^n} u_t(u_{tt} - c^2\nabla\cdot\nabla u)dx. \tag{36}$$

Noting that $u_{tt} - c^2\nabla\cdot\nabla u = u_{tt} - c^2\triangle u = 0$, by rearrangement of the wave equation, implies that $\frac{dE}{dt} = 0$. Thus any phenomenon modeled by the wave equation must conserve energy.

(2) **The fundamental vibrational mode of a standing wave is the most energetic.** This is a direct consequence of results from Fourier series. For a standing wave with fixed endpoints wave we have the solution,[13]

$$u(x,t) = \sum_{n=1}^{\infty} B_n\cos(\sqrt{\lambda_n}ct)\sin(\sqrt{\lambda_n}x). \tag{37}$$

[9] See problems 24-29 from section 12.9 in the text page 586.

[10] Noting that this equation is linear but has variable coefficients implies that we NEED to use power series methods to solve this problem.

[11] This makes solving the boundary condition $W(\sqrt{\lambda}R) = 0$ intractable by hand and requires numerical root approximation techniques applied to the series definition.

[12] Here I consider any arbitrary finite-dimension. In class we made concrete statements about the $\mathbb{R}^{1+1}$ case.

[13] For simplicity we have assumed that the vibrations are set into motion by an initial deformation with no initial velocity. If this is not the case then we can make the same statement but it is not as clean.

Going back to Fourier series we found that Fourier coefficients have the form, $B_n \propto n^{-\alpha}$ where $\alpha \in \mathbb{N}$ and thus $|B_{n+1}| < |B_n|$ for all n.[14] This implies that $|B_1|$ is the largest Fourier coefficient and since the amplitude of the Fourier mode controls the energy of the Fourier mode the first mode is the most energetic and for these reasons it is called *fundamental*. [15]

(3) **Traveling wave solutions of the wave equation propagate with finite speed.** It can be shown that the general solution to the wave equation in $\mathbb{R}^{1+1}$ can be written in the form,

$$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(y)dy, \tag{38}$$

which expresses solutions to the wave equation as right and left traveling waves of speed c. If we consider the propagation of electromagnetic waves modeled by the wave equation then c is the speed of light. If we consider the propagation of acoustic waves in the air at sea level then c corresponds to the mach number. The most important feature is that $c < \infty$ and we conclude that all evolution controlled by the wave equation has a finite speed of propagation.

5. Lecture Goals

Our goals with this material will be:

- Compare and contrast the physics of the heat and wave equations and the behaviors of their corresponding solutions.
- Understand how the spatial complexity of the PDE changes when considering different domains.

6. Lecture Objectives

The objectives of these lessons will be:

- Solve the wave equation on bounded domains in both $\mathbb{R}$ and $\mathbb{R}^2$ by the use of Fourier series.
- Discuss the solution to the wave equation on a circular geometry and introduce Bessel functions.
- Characterize standing and traveling solutions of the wave equation.

End Quote of The Semester	
You know that I care what happens to you, and I know that you care for me too. So I don't feel alone, or the weight of the stone, now that I've found somewhere safe to bury my bone. And any fool knows a dog needs a home, a shelter from pigs on the wing.	
	Pink Floyd : Pigs on the Wing - Part 2 (1977)

[14]Formally, this is known as the *Riemann-Lebesgue lemma*.

[15]This is a name coming from acoustics, where one concludes that the first mode carries the majority of sound.