Calculation of retarded fields from potentials.

- method used for current, charge density sources.

retarded potentials

$$\Phi(\vec{r},t) = \int_V \frac{\rho(\vec{r}\,', t - R/c)}{R}\, d^3r' \qquad w/\ R = |\vec{r} - \vec{r}\,'|$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int_V \frac{J(\vec{r}\,', t - R/c)}{R}\, d^3r'$$

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla\times\vec{A}$$

do calc for $\vec{E}$:

$$\frac{\partial \vec{A}}{\partial t} = \frac{1}{c}\int \frac{\partial_t J(\quad)}{R}\, d^3r' \qquad \text{simple } t\text{-dependence}$$

$$\nabla\Phi = \int \left(\rho(\ )\,\nabla\left(\frac{1}{R}\right) + \frac{1}{R}\nabla\rho \right) d^3r'$$

$$\nabla\left(\frac{1}{R}\right) = -\frac{1}{R^2}\hat{R} \qquad \hat{R} = \frac{\vec{r} - \vec{r}\,'}{R}$$

$$\nabla\rho(r', t - R/c) = \left.\frac{\partial\rho}{\partial t}\right|_{t_r} \nabla\left(t - \frac{R}{c}\right) = -\frac{1}{c}\left.\frac{\partial\rho}{\partial t}\right|_{t_r}\hat{R}$$

together:

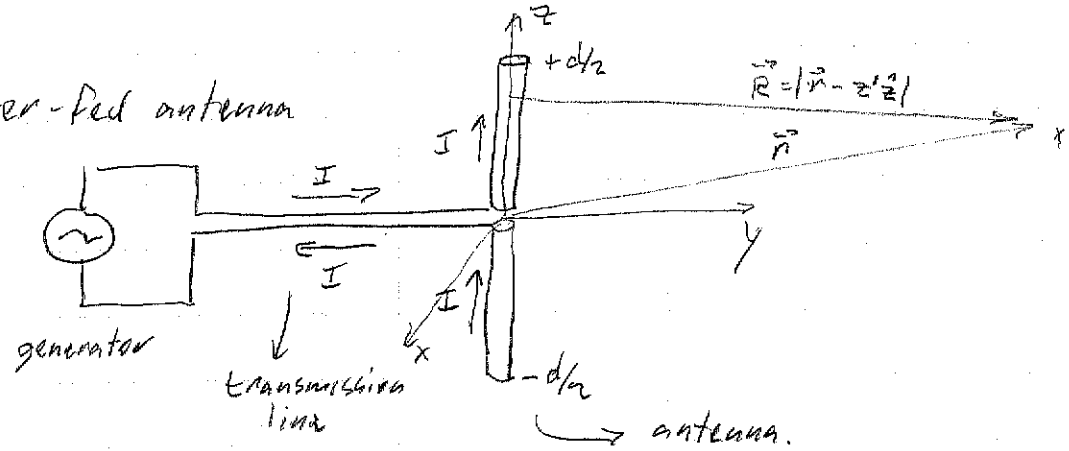
$$\vec{E} = \int_V \left(\frac{[\rho]}{R^2}\hat{R} + \frac{[\partial_t\rho]}{cR}\hat{R} - \frac{[\partial_t\vec{J}]}{c^2R} \right) d^3r' \qquad [\ ] \text{ means evaluate at } t_r = t - R/c$$

Similarly:

$$\vec{B} = \int_V \left(\frac{[\vec{J}]\times\hat{R}}{cR^2} + \frac{[\partial_t\vec{J}]\times\hat{R}}{c^2R} \right) d^3r'$$

Application: linear antenna (HM 9-4)

center-fed antenna



model current: oscillating, standing wave along antenna.

$$I(z,t) = I_0 e^{-i\omega t} \sin k\left(\frac{d}{2} - |z|\right)$$

at gap $I = I_0 \sin(kd/2)\, e^{i\omega t}$

current density:

$$J(z,t) = I(z,t)\, \delta(x)\, \delta(y)\, \hat{z} \qquad \text{(small cross-section)}$$

We will put this into the integral over x', y', z'

There are two ways to do these calculations:

1) find $\vec{A}$, then calculate $\vec{B}$, $\vec{E}$

2) find $\vec{B}$ directly.

method 1

$$\vec{A} = \frac{1}{c} \int_V \frac{\vec{J}(\vec{r}', t - R/c)}{R}\, d^3r'$$

$$= \frac{1}{c} \int \hat{z} I_0 e^{-i\omega(t - R/c)} \frac{\sin k\left(\frac{d}{2} - |z'|\right)}{R}\, dz' \underbrace{\int \delta(x)dx \int \delta(y)dy}_{\text{integrate to } 1}$$

we're now integrating on z' only

note $R \equiv |\vec{r}' - \vec{r}| = |\vec{r} - z'\hat{z}|$

$$\vec{A} = \hat{z} \frac{I_0 e^{-i\omega t}}{c} \int \frac{\sin k\left(\frac{d}{2} - |z'|\right)}{|\vec{r} - \vec{r}'|}\, e^{ik_0|\vec{r} - \vec{r}'|}\, dz'$$

Dipole radiation (small linear antenna)

Approximations to R: $\quad r >> d$

in denominator, safe to let $R \to r$

in $e^{ik_0|\vec{r}-\vec{r}'|}$ we must be more careful b/c of phase variations:

law of cosines:

$$|\vec{r}-\vec{r}'| = \sqrt{r^2 - 2rr'\cos\theta + r'^2}$$

or, if $\hat{n} = \hat{r}$, $\quad r'\cos\theta = \hat{n}\cdot\vec{r}'$

$$R = r\sqrt{1 - 2\frac{\hat{n}\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2}$$

$$\simeq r\left(1 - \frac{\hat{n}\cdot\vec{r}'}{r} + \frac{1}{2}\left(\frac{r'}{r}\right)^2 - \frac{1}{8}\left(\frac{2\hat{n}\cdot\vec{r}'}{r}\right)^2 + \dots\right)$$

$$= r\left(1 - \frac{r'\cos\theta}{r} + \frac{1}{2}\left(\frac{r'}{r}\right)^2\sin^2\theta + \dots\right)$$

in exponent, we have kR

we can neglect terms in expansion when they are $<< 2\pi$

$\therefore$ neglect third term if

$$k_0 r\frac{1}{2}\left(\frac{r'}{r}\right)^2\sin^2\theta << 2\pi$$

max value of: $\sin^2\theta = 1$

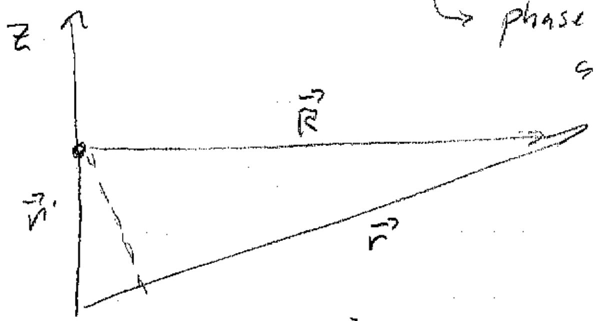
$r' = d/2$

$$\therefore \frac{2\pi}{\lambda}\cdot\frac{1}{2}\frac{d^2}{4}\frac{1}{r} << 2\pi$$

or $\underline{r >> d^3/8\lambda}$ Fraunhofer limit observe:

evaluation at the retarded time for oscillating source

$\rightarrow d\vec{A} \sim e^{-i\omega t} \underbrace{e^{i\phi(\vec{r}')}}$ → phase shift in arrival from each source point.



to do calculation of $\vec{A}$ we must make approximations to

$$|\vec{r} - \vec{r}'| = \sqrt{r^2 + z'^2 - 2rz'\cos\theta}$$

for $r \gg d/2$ (far field)

$$\vec{A} \sim \hat{z} A_0 \frac{e^{i(k_0 y - \omega t)}}{y} \qquad \text{along } y\text{-axis}$$

(will do this in more detail later)

method 2 calculate $\vec{B}$ directly.

two terms:
$$\frac{[J]}{cR^2} \rightarrow \frac{e^{-i\omega t}}{R} \frac{I_0 \sin(\,) e^{ik_0 R}}{cR}$$
$$\frac{[\partial_t J]}{c^2 R} \rightarrow \frac{-i\omega}{c} \frac{I_0 \sin(\,) e^{ik_0 R}}{cR}$$

for $1/R \ll \frac{\omega}{c} = k_0$ $\quad (R \gg \lambda_0)$

keep second term only