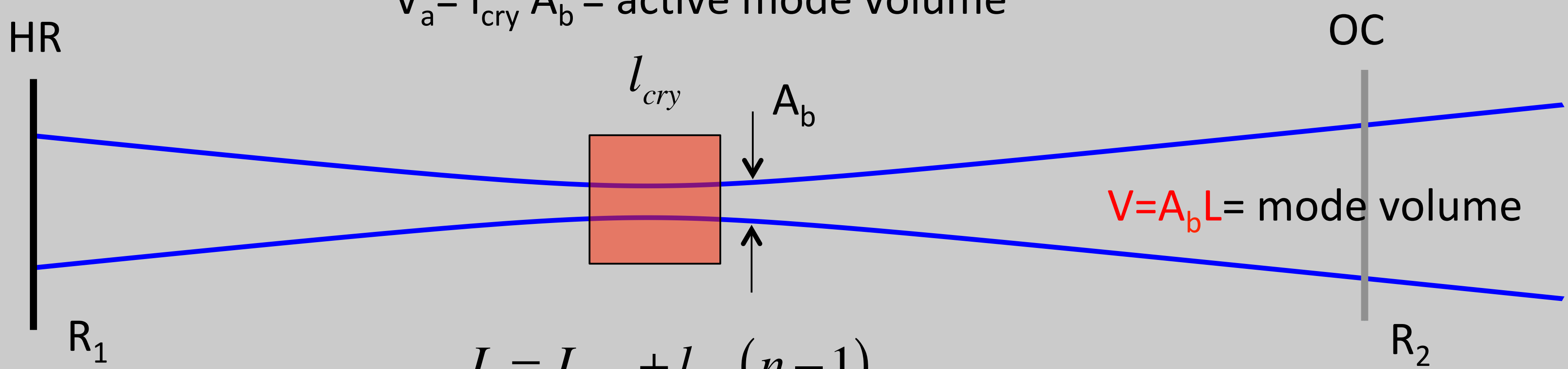
Model laser cavity

$V_a = l_{cry} A_b$ = active mode volume



$$L = L_{cav} + l_{cry}(n-1)$$

$\mathcal{L}_i$ Internal passive losses

$$T_{RT} = \frac{2L}{c}$$ Round-trip time

$$\phi = \frac{\rho V}{h\nu_{21}}$$ Total number of photons in cavity

$$\rho = \frac{2I}{c}$$ Energy density. 2 I = average intensity (both directions), neglecting standing wave interference

$$\phi = \frac{2I}{c}\frac{L A_b}{h\nu_{21}} = I\frac{A_b T_{RT}}{h\nu_{21}}$$

Laser dynamics equations, initial conditions

- Coupled equations for photon number and inversion density

$$\frac{d\phi}{dt} = V_a B N_2 (\phi + 1) - \frac{\phi}{\tau_c}$$

$$\frac{dN_2}{dt} = R_P - B\phi N_2 - N_2 / \tau_{21}$$

Initial conditions (t=0)

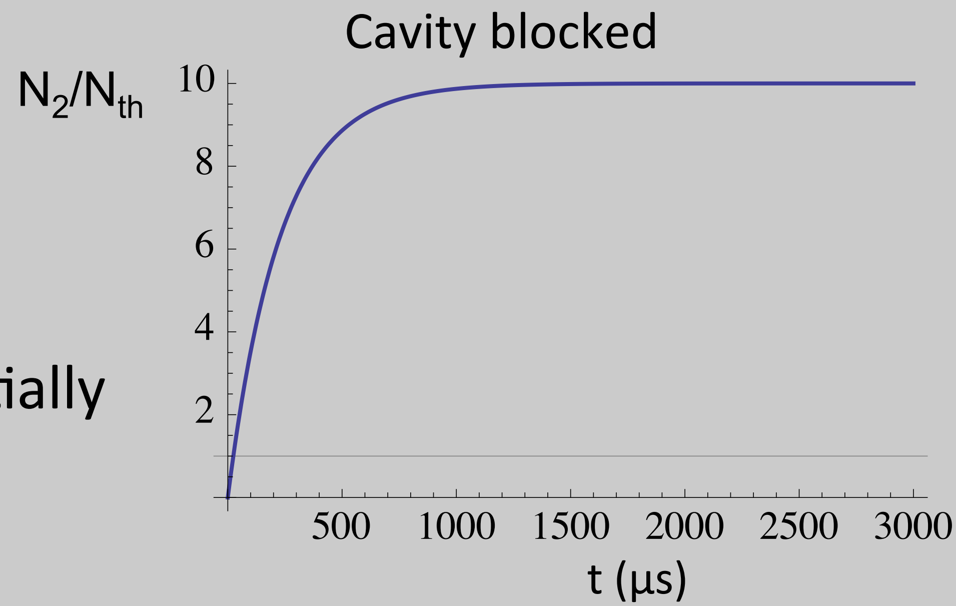
- 'cold' cavity (gain switching, relaxation oscillations)
  - R_p was at 0 (no pump), the pump turns on
  - N_2 starts at 0
  - No lasing, but ϕ=1, representing 1 photon populating the laser mode (QED vacuum fluctuations that stimulate spontaneous emission)
- Q-switching
  - Rp on before t<0, N2 in equilibrium, but no lasing

Laser start-up dynamics

- "cold" cavity: pump turns on $\phi = 0$

$$\frac{dN_2}{dt} = R_P - N_2 / \tau_{21}$$

$$N_2(t) = R_P \tau_{21} \left(1 - e^{-t/\tau_{21}}\right)$$



- Once threshold is reached

  φ starts to build up exponentially

$$\frac{d\phi}{dt} = V_a B N_2 \phi - \frac{\phi}{\tau_c}$$

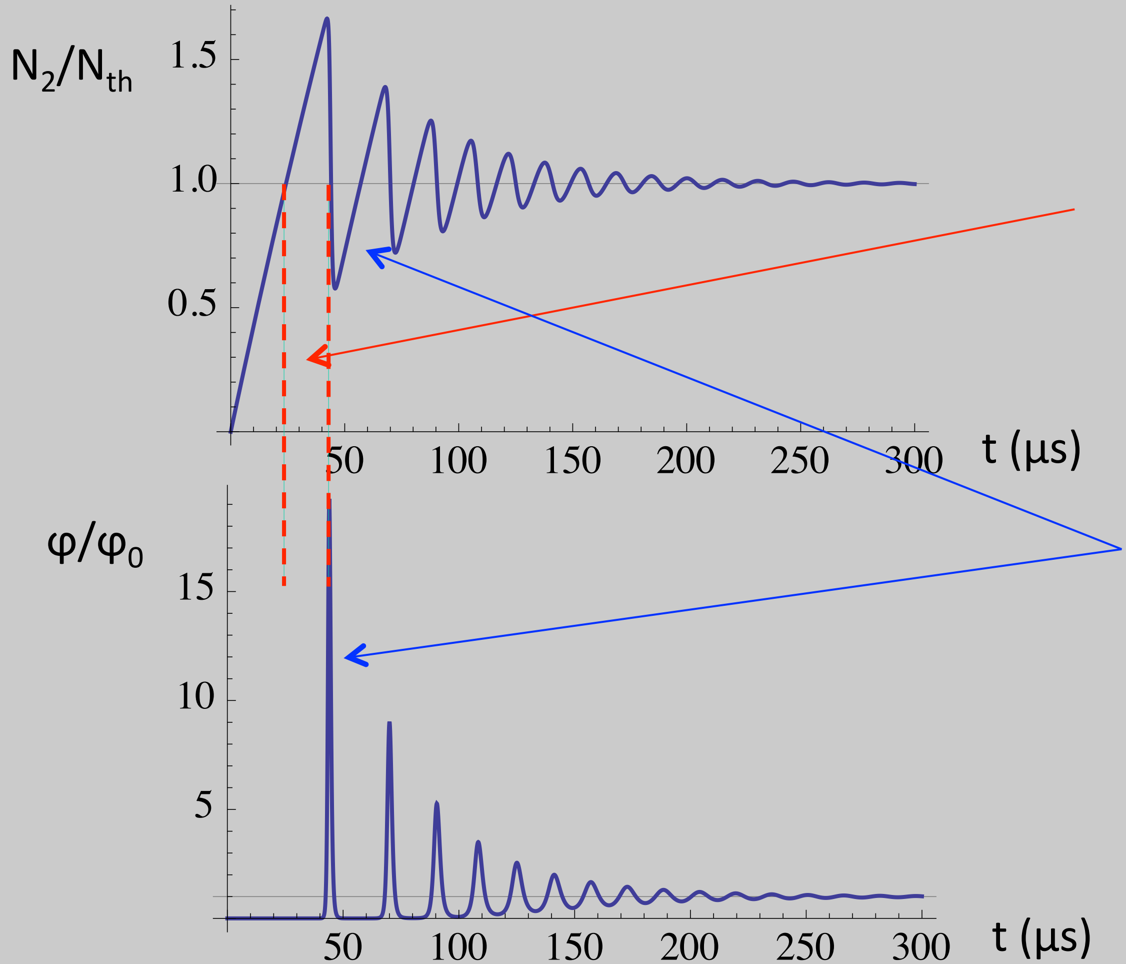
  – For operation just above threshold, let

$$N_2 = N_{th} + \delta N_2 \qquad N_{th} = \frac{\gamma}{\sigma l_{cry}} \qquad \tau_c = L/\gamma c$$

$$\frac{d\phi}{dt} = V_a B \left(N_{th} + \delta N_2\right)\phi - \frac{\phi}{\tau_c}$$

Relaxation oscillations

- Spiking of laser output before stabilization



Lag time for photon build-up allows pumping to go above threshold

Laser power spikes high and depletes the stored energy, terminates lasing

Excited state population builds up again, starting at a higher level

Small-signal analysis of relaxation oscillations

- **Inversion density:** start from time dependent equation

$$\frac{dN_2}{dt} = R_P - B\phi N_2 - N_2 / \tau_{21}$$

- Assume small departure from equilibrium, constant R_p

$$\frac{dN'}{dt} = R_P - B\left(\phi_0 + \phi'\right)\left(N_0 + N'\right) - \left(N_0 + N'\right)/\tau_{21}$$

$$\frac{dN'}{dt} = R_P - B\left(\phi_0 N_0 + \phi' N_0 + \phi_0 N' + \phi' N'\right) - \left(N_0 + N'\right)/\tau_{21}$$

  - Neglect products of small terms
  - Equilibrium values add to zero

$$\frac{dN'}{dt} = -\left(B\phi_0 + \frac{1}{\tau_{21}}\right)N' - B\phi' N_0$$

Small-signal analysis of relaxation oscillations

- **Photon number:** start from time dependent equation

$$\frac{d\phi}{dt} = V_a B N_2 \phi - \frac{\phi}{\tau_c}$$

- Assume small departure from equilibrium, constant R_p

$$\frac{d\phi'}{dt} = V_a B\left(N_0 + N'\right)\left(\phi_0 + \phi'\right) - \frac{\left(\phi_0 + \phi'\right)}{\tau_c}$$

$$\frac{d\phi'}{dt} = V_a B\left(N_0\phi_0 + N'\phi_0 + N_0\phi' + N'\phi'\right) - \frac{\left(\phi_0 + \phi'\right)}{\tau_c}$$

  - Neglect products of small terms
  - Equilibrium values add to zero

$$\frac{d\phi'}{dt} = \left(V_a B N_0 - \frac{1}{\tau_c}\right)\phi' + V_a B N' \phi_0$$

$$\boxed{\frac{d\phi'}{dt} = V_a B N' \phi_0}$$

Coupled equations: damped SHO

- Convert two 1st order eqns to one 2nd order

$$\frac{dN'}{dt} = -\left(B\phi_0 + \frac{1}{\tau_{21}}\right)N' - B\phi' N_0$$

$$\frac{d^2N'}{dt^2} = -\left(B\phi_0 + \frac{1}{\tau_{21}}\right)\frac{dN'}{dt} - B\frac{d\phi'}{dt}N_0$$

$$\frac{d\phi'}{dt} = V_a B N' \phi_0$$

$$\frac{d^2N'}{dt^2} + \left(B\phi_0 + \frac{1}{\tau_{21}}\right)\frac{dN'}{dt} + B\left(V_a B N' \phi_0\right)N_0 = 0$$

$$V_a B N_0 - \frac{1}{\tau_c} = 0$$

$$\frac{d^2N'}{dt^2} + \left(B\phi_0 + \frac{1}{\tau_{21}}\right)\frac{dN'}{dt} + \frac{B\phi_0}{\tau_c}N' = 0$$

Relaxation oscillation solution

- Compare to standard SHO equation:

$$\frac{d^2N'}{dt^2}+\left(B\phi_0+\frac{1}{\tau_{21}}\right)\frac{dN'}{dt}+\frac{B\phi_0}{\tau_c}N'=0 \qquad \frac{d^2N'}{dt^2}+\frac{2}{t_0}\frac{dN'}{dt}+\Omega_0^2N'=0$$

- Expect exponential solutions $N'(t)=N'_0e^{pt}$

$$p^2+\frac{2}{t_0}p+\Omega_0^2=0 \qquad \rightarrow p=-\frac{1}{t_0}\pm\sqrt{\frac{1}{t_0{}^2}-\Omega_0^2}$$

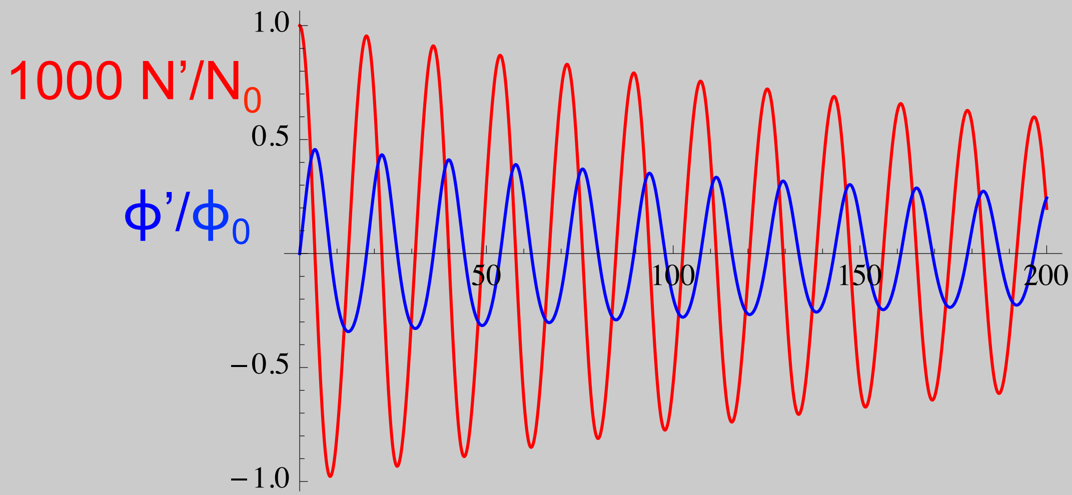
- Oscillatory solutions if

$$\Omega_0>\frac{1}{t_0} \qquad \rightarrow \Omega=\sqrt{\Omega_0^2-\frac{1}{t_0{}^2}}$$

$$N'(t)=N'_0e^{-t/t_0}\cos(\Omega t+\beta) \qquad \phi'(t)=\phi'_0e^{-t/t_0}\sin(\Omega t+\beta)$$

Dynamic solutions

- Change in N leads response in ϕ



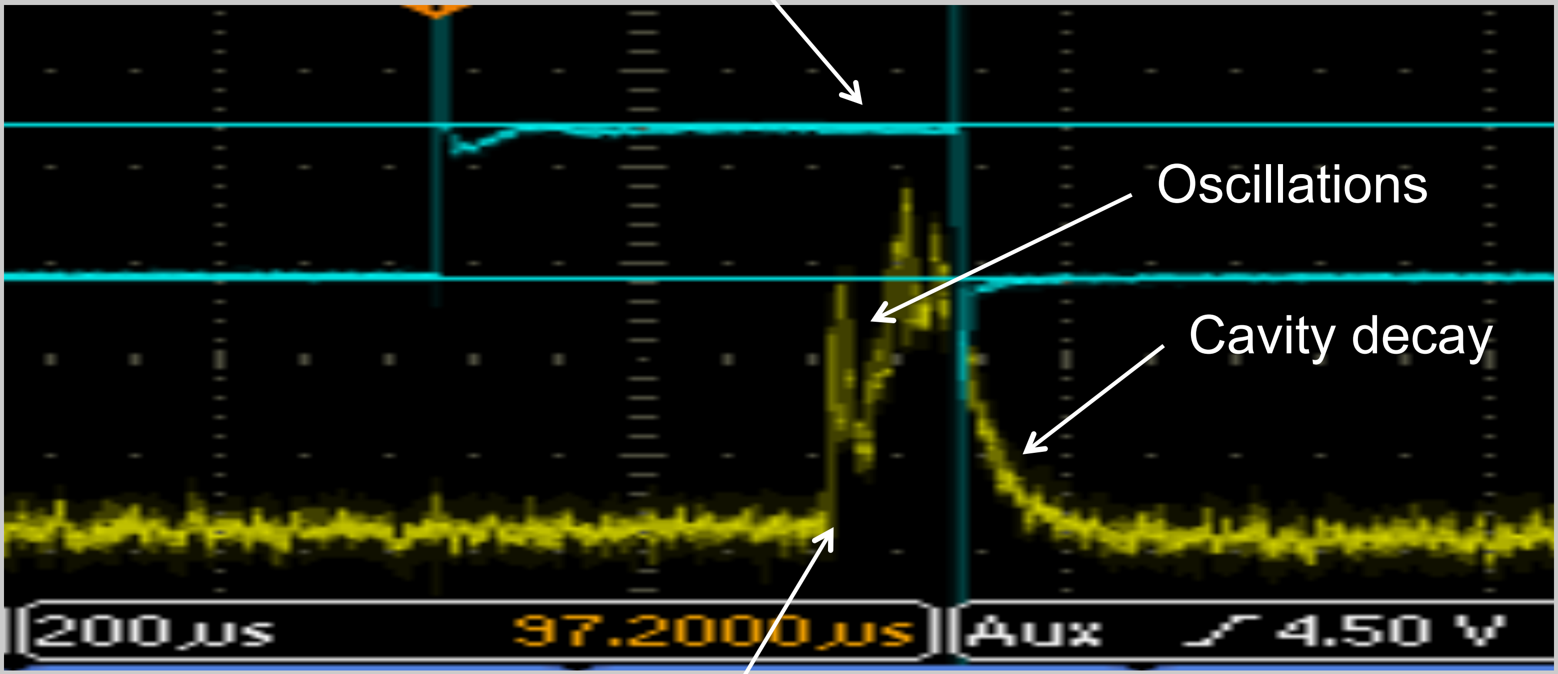
- Damping timescale depends on pumping level

$$t_0 = 2\tau_{21} / x \qquad x = P_p / P_{th}$$

- Oscillation frequency $\Omega_0 = \dfrac{x-1}{\tau_c \tau_{21}}$
  - Typically no oscillations, spiking in gas lasers
  - Ripple in pump can drive oscillation

Relaxation oscillations in a gain-switched CeNd:YAG laser

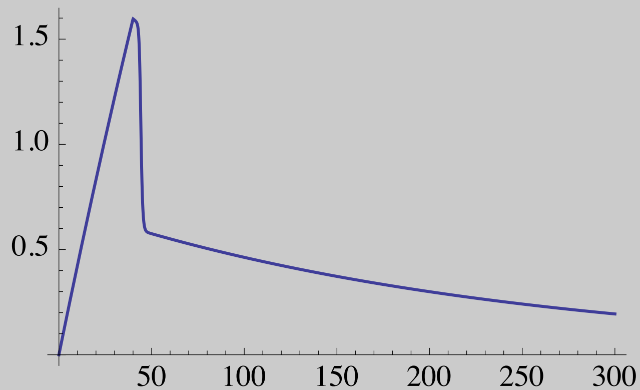
- Square pump pulse



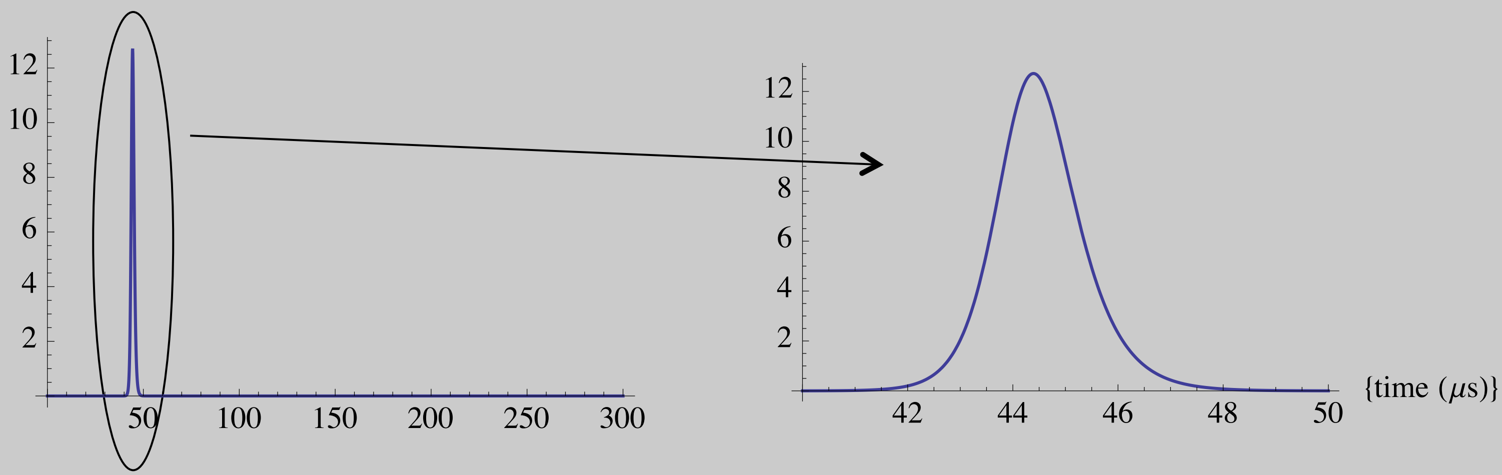
Transient behavior can lead to quantitative information about laser

Gain switching: controlled relaxation oscillation

- Pulse laser by stopping pump after first initial spike



Output pulse is short, but μs scale

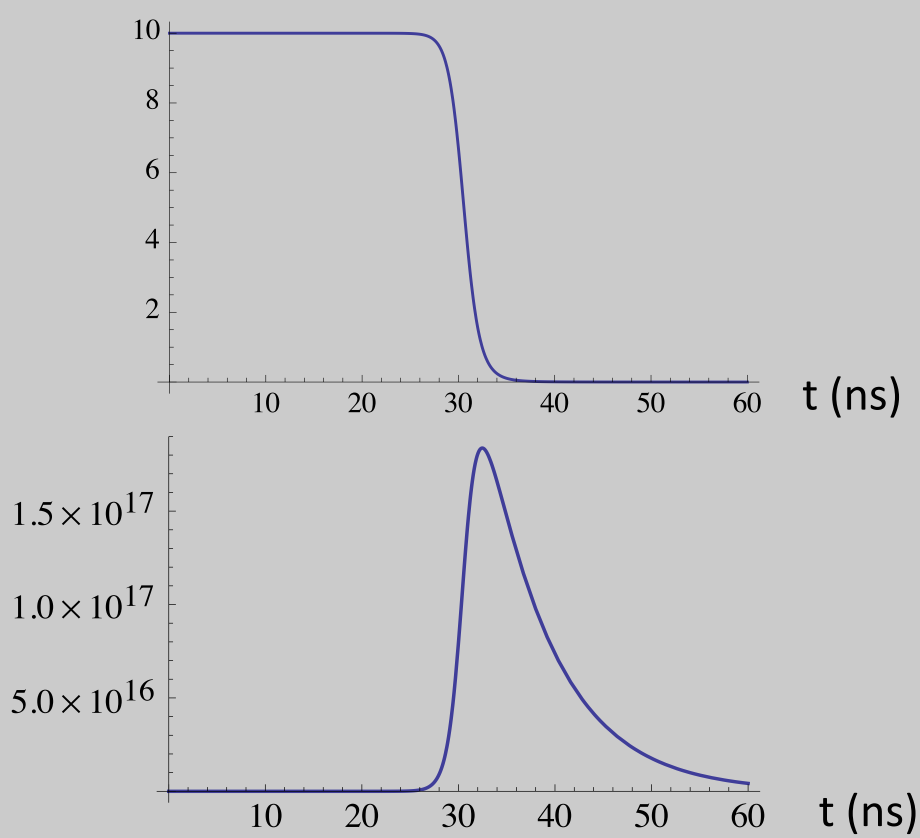


Q-switching

- Use an additional component to hold off lasing to allow build up of stored energy
  - Inversion density can reach levels much higher than threshold
- Active q-switch: can trigger externally
  - Electro-optic, Acousto-optic
- Passive q-switch: cheaper,
  - Saturable absorber, e.g. dye, Cr:YAG, …
- Pumping:
  - Pulsed pump: deliver pump energy in $t < t_{21}$
  - CW pump: repetitive Q-switching, high reprate. Optimization for average power vs peak pulse energy

Q-switching dynamics

- Start with high inversion density
- Fast opening of switch



Output pulse is orders of magnitude shorter duration than gain switching.

Leading edge duration:
- Gain controls build up time
- Hold-off of buildup allows the gain to reach high values

Trailing edge duration:
- Saturation and cavity loss

Electro-optic q-switch

- High voltage on nonlinear crystal controls birefringence: 0 to quarter-wave

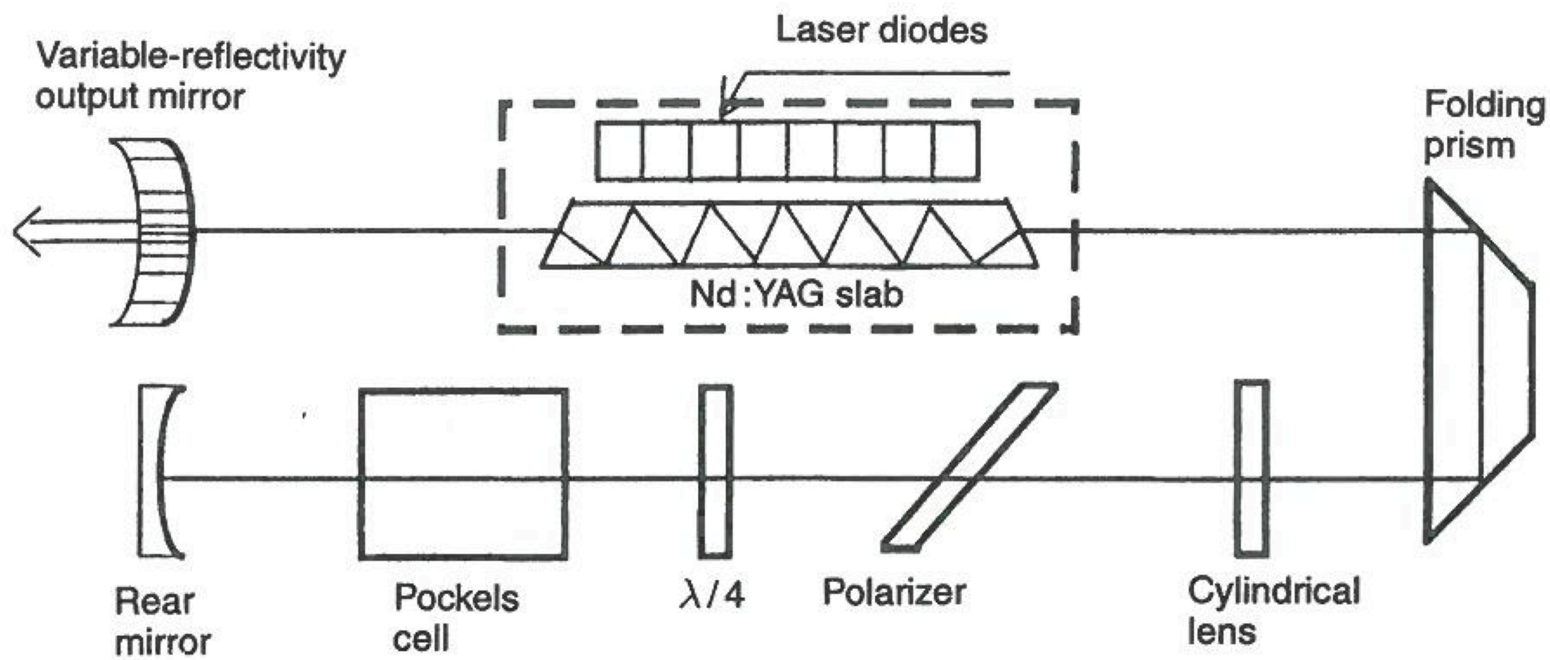
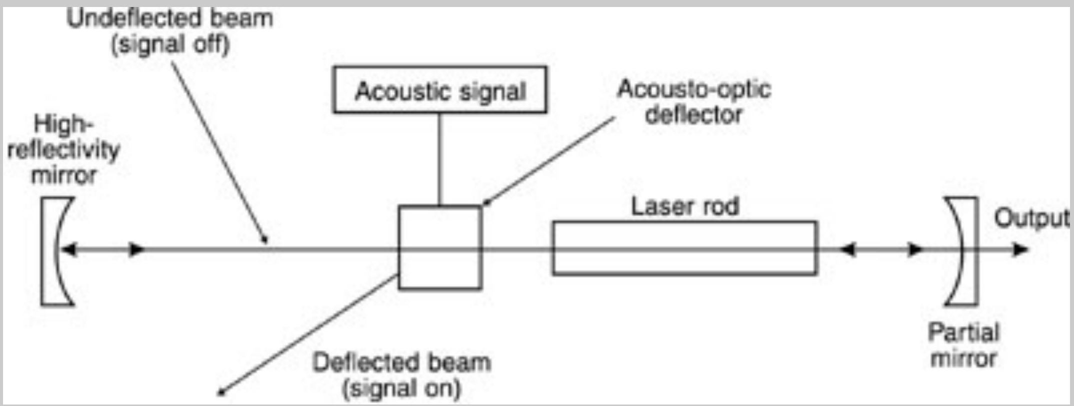


Fig. 5.50. Diode-pumped Nd:YAG slab laser with positive-branch unstable resonator and variable reflectivity output coupler [5.76]

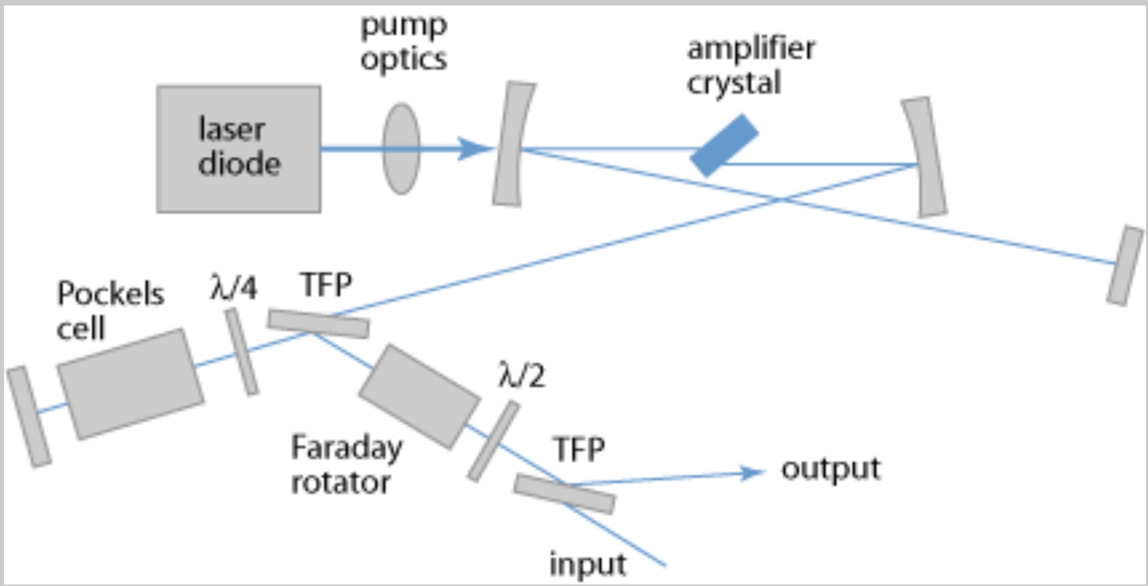
Acousto-optic q-switch

- RF transducer launches high amplitude sound wave in crystal
- Wave acts as a grating and scatters light out



Cavity-dumping

- No partially transmitting mirror – polarization control only



Fast Q-switching dynamics: pumping

$$\frac{d\phi}{dt} = V_a B N_2 \phi - \frac{\phi}{\tau_c} \qquad \frac{dN_2}{dt} = R_P - B\phi N_2 - N_2 / \tau_{21}$$

- We can separate steps since timescales are different
- Pump phase (Q-sw closed) $\phi_0 = 0$

$$\frac{dN_2}{dt} = R_P - N_2 / \tau_{21}$$

- Pump pulse duration << fluorescence time

$$\frac{dN_2}{dt} = R_P(t) \rightarrow N_2(t) = \int_0^t R_p(t')dt' = \frac{E_p}{V_a h\nu_p} \qquad N_{init} = \frac{E_p}{V_a h\nu_p}$$

- After pumping, inversion is *below* threshold for *closed* cavity
- Inversion is *above* threshold for *open* cavity

$$X_{pump} = \frac{N_{init}}{N_{th}} = \frac{E_p}{E_{cr}} \qquad N_{th} = \frac{\gamma}{\sigma_{21} l_{cry}}$$

Fast Q-switching dynamics: build-up

- Q-sw opens, photon number can accumulate
  - For sufficient gain, build up is faster than fluor time
  - Before saturation, N_2 is steady = N_{init}

$$X_{pump} = \frac{N_{init}}{N_{th}}$$

$$\frac{d\phi}{dt} = V_a B\, N_{init}\, \phi - \frac{\phi}{\tau_c} = \left(V_a B\, N_{init} - \frac{1}{\tau_c} \right) \phi$$

$$\tau_c = L/\gamma\, c$$

$$\frac{d\phi}{dt} = \left(V_a B\, X_{pump} N_{th} - \frac{1}{\tau_c} \right) \phi = \left(V_a \frac{\sigma_{21} c}{V} X_{pump} \frac{\gamma}{\sigma_{21} l_{cry}} - \frac{1}{\tau_c} \right) \phi$$

$$N_{th} = \frac{\gamma}{\sigma_{21} l_{cry}}$$

$$B = \frac{\sigma_{21} c}{V}$$

$$\frac{d\phi}{dt} = \left(\frac{l_{cry}}{L} c\, X_{pump} \frac{\gamma}{l_{cry}} - \frac{1}{\tau_c} \right) \phi = \frac{\phi}{\tau_c} \left(X_{pump} - 1 \right)$$

$$\phi(t) = \exp\left[\frac{X_{pump} - 1}{\tau_c} t \right]$$

Exponential growth during build-up

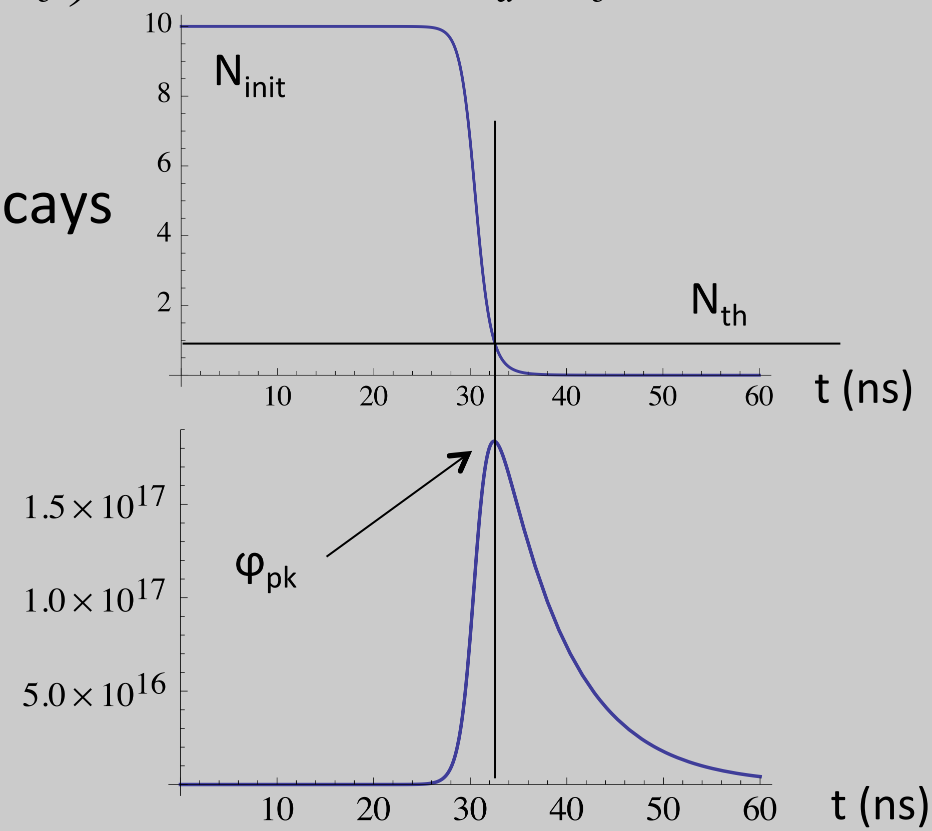
Fast Q-switching dynamics: pulse peak

- At peak of pulse, slope of pulse shape = 0

$$\frac{d\phi}{dt} = V_a B N_2 \phi - \frac{\phi}{\tau_c} = \left(V_a B N_2 - \frac{1}{\tau_c} \right) \phi = 0 \rightarrow N_2 = \frac{1}{V_a B \tau_c} = N_{th}$$

  - N_2 has dropped down to N_{th}
- After peak, photon number decays
  - N_2=N_{fin}
  - N_{fin}~0 if enough saturation

$$\frac{d\phi}{dt} = -\frac{\phi}{\tau_c} \rightarrow \phi(t) \sim \phi_{pk} \exp\left[-t/\tau_c\right]$$



Peak photon number in Q-switched pulse

- Trick to get ϕ_{pk}:
  - if we neglect fluorescence, ϕ and N_2 are connected
  - Also negect pumping during pulse duration

$$\frac{d\phi}{dN_2} = \frac{d\phi/dt}{dN_2/dt} = \frac{V_a B N_2 \phi - \phi/\tau_c}{-B\phi N_2} = -V_a + \frac{1}{B\tau_c N_2}$$

  - Integrate from initial to peak

$$\phi_{pk} - 0 = \int_{N_{init}}^{N_{th}} \left(-V_a + \frac{1}{B\tau_c N_2} \right) dN_2 = V_a \left(N_{init} - N_{th} \right) - \frac{1}{B\tau_c} \ln\left[\frac{N_{init}}{N_{th}} \right]$$

$$\frac{\phi_{pk}}{V_a} = N_{init} - N_{th} - N_{th} \ln\left[X_{pump} \right]$$

$$X_{pump} = \frac{N_{init}}{N_{th}}$$

= photons extracted - photons lost during build-up

Peak output power

- Output power is proportional to photon number

$$P_{out}(t) = \frac{\gamma_2 h\nu}{T_{RT}} \phi(t)$$

$$P_{pk} = \frac{\gamma_2 h\nu}{T_{RT}} \phi_{pk} = \frac{\gamma_2 h\nu}{T_{RT}} V_a \left(N_{init} - N_{th} - N_{th} \ln\left[X_{pump} \right] \right)$$

$$P_{pk} = \frac{\gamma_2 h\nu}{T_{RT}} V_a N_{init} \left(1 - \frac{N_{th}}{N_{init}} - \frac{N_{th}}{N_{init}} \ln\left[X_{pump} \right] \right)$$

$$X_{pump} = \frac{N_{init}}{N_{th}}$$

$$P_{pk} = \frac{\gamma_2}{T_{RT}} E_{init} \left(1 - \frac{1}{X_{pump}} \left(1 + \ln\left[X_{pump} \right] \right) \right)$$

Total photon number

- Integrate over history of pulse

$$\frac{d\phi}{dt} = V_a B N_2 \phi - \frac{\phi}{\tau_c} \quad \rightarrow \int d\phi = \phi(\infty) - \phi(0) = 0$$

$$\int \left(V_a B N_2 \phi - \frac{\phi}{\tau_c} \right) dt = V_a \int B N_2 \phi \, dt - \frac{1}{\tau_c} \int \phi \, dt = 0$$

- Get qty in red from N_2 equation:

$$\frac{dN_2}{dt} = -B \phi N_2 \rightarrow N_{init} - N_{fin} = \int_0^\infty B \phi N_2 \, dt$$

$$\int \phi \, dt = \tau_c V_a \int B N_2 \phi \, dt = \tau_c V_a \left(N_{init} - N_{fin} \right)$$

N_{fin} = inversion density left after pulse

Q-switched output energy

- Output energy from OC results from integration over total photon number

$$E_{out} = \int P_{out}(t)\,dt = \frac{\gamma_2 h\nu}{T_{RT}} \int \phi(t)\,dt$$

$$\int \phi\, dt = \tau_c V_a \left(N_{init} - N_{fin}\right)$$

$$E_{out} = \gamma_2 h\nu \frac{c}{2L} \tau_c V_a \left(N_{init} - N_{fin}\right)$$

$$\tau_c = L/\gamma c$$

$$E_{out} = \gamma_2 h\nu \frac{c}{2L} \frac{L}{\gamma c} V_a \left(N_{init} - N_{fin}\right) = \frac{\gamma_2}{2\gamma} h\nu V_a \left(N_{init} - N_{fin}\right)$$

Output coupling/total losses

Total energy extracted

$$E_{out} = \frac{\gamma_2}{2\gamma} h\nu V_a N_{init} \left(\frac{N_{init} - N_{fin}}{N_{init}} \right) \equiv \frac{\gamma_2}{2\gamma} E_{init} \eta_E$$

η_E =Extraction efficiency

Q-switching extraction efficiency

- When pulse finishes, some stored energy can be left
- To get final inv density, integrate ϕ(N) from initial to final

$$\int \phi(N)\,dN = 0 - 0 = V_a\left(N_{init} - N_{fin}\right) - \frac{1}{B\tau_c}\ln\left[\frac{N_{init}}{N_{fin}}\right]$$

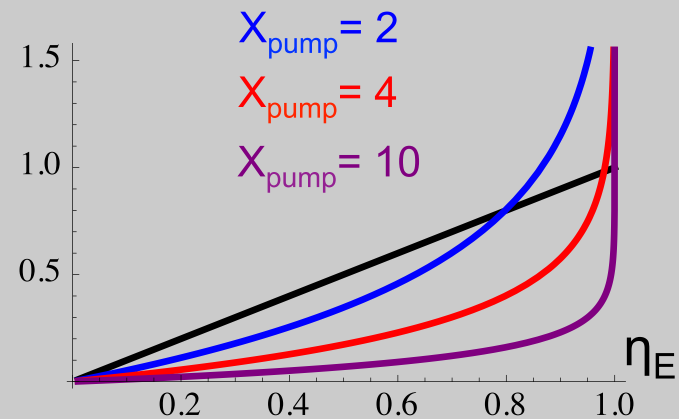
– Extraction efficiency:

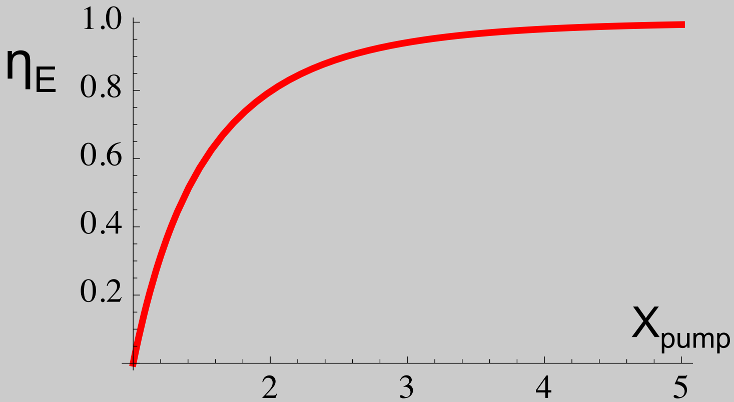
$$\eta_E = \frac{N_{init} - N_{fin}}{N_{init}} = \frac{1}{B\tau_c V_a N_{init}}\ln\left[\frac{N_{init}}{N_{fin}}\right] = \frac{N_{th}}{N_{init}}\ln\left[\frac{N_{init}}{N_{fin}}\right]$$

$$\eta_E = 1 - \frac{N_f}{N_{pk}} \rightarrow \frac{N_{pk}}{N_f} = \frac{1}{1-\eta_E}$$

$$X_{pump} = \frac{N_{init}}{N_{th}}$$

$$\eta_E = \frac{1}{X_{pump}}\ln\left[\frac{1}{1-\eta_E}\right]$$





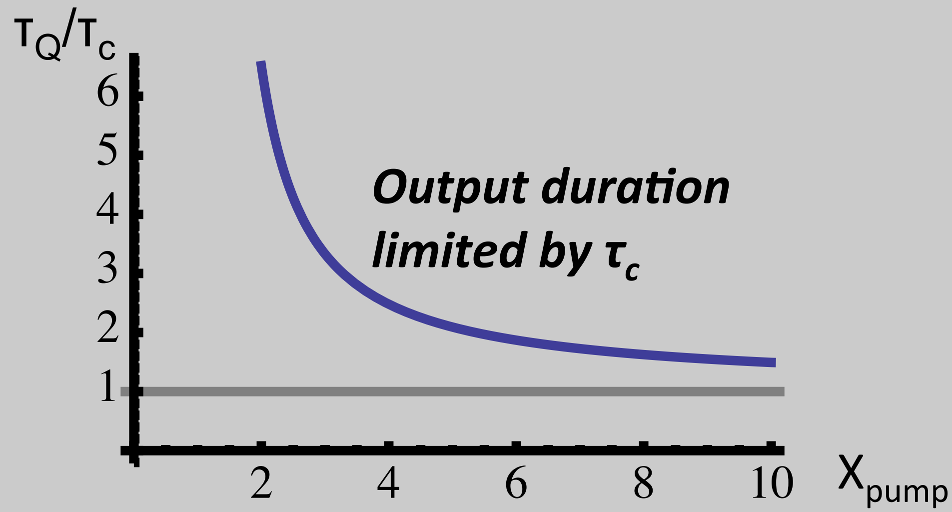
Estimate Q-switched pulse duration

- From the output energy and the peak power, we can estimate the duration of the pulse

$$\tau_Q \approx \frac{E_{out}}{P_{pk}} = \frac{\frac{\gamma_2}{2\gamma} E_{init} \eta_E}{\frac{\gamma_2}{T_{RT}} E_{init} \left(1 - \frac{1}{X_{pump}} \left(1 + \ln\left[X_{pump} \right] \right) \right)}$$

$$\tau_c = \frac{L}{\gamma c} = \frac{T_{RT}}{2\gamma}$$

$$\tau_Q \approx \tau_c \frac{\eta_E}{1 - \frac{1}{X_{pump}} \left(1 + \ln\left[X_{pump} \right] \right)}$$



Example: Q-switched microchip laser

Diode-pumped microchip lasers electro-optically Q switched at high pulse repetition rates

September 1, 1992 / Vol. 17, No. 17 / OPTICS LETTERS 1201

J. J. Zayhowski and C. Dill III

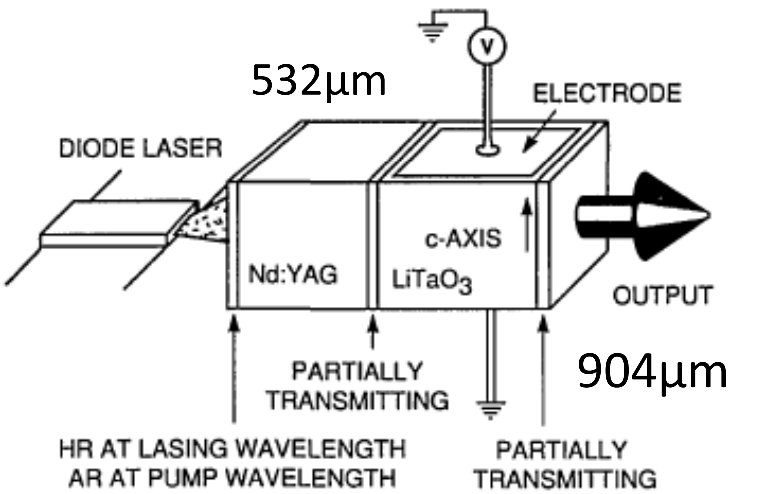


Fig. 1. Illustration of an electro-optically Q-switched microchip laser. HR, highly reflecting; AR, antireflecting.

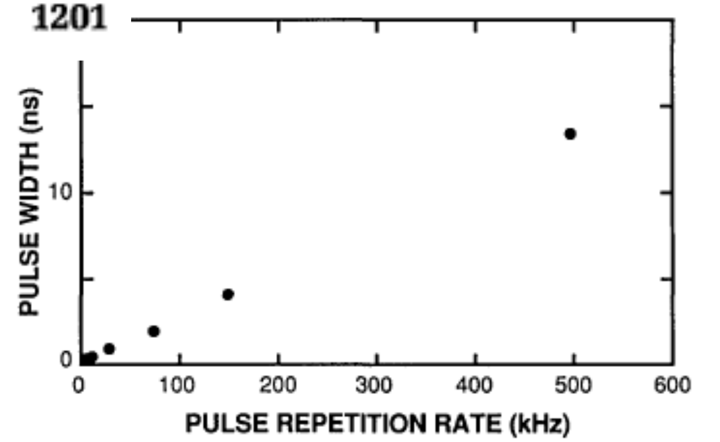


Table 1. Characteristics of the Output Pulses from a 1.064-μm, Diode-Pumped, Electro-Optically Q-Switched Nd:YAG Microchip Laser

Pulse Rate (kHz)	Pulse Width (ns)[a]	Time-Averaged Power (mW)	Pulse Energy (μJ)[b]	Peak Power (W)[c]
5	0.27	34	6.80	25,185
10	0.43	50	5.00	11,627
30	0.91	53	1.77	1941
75	2.0	55	0.73	367
150	4.1	57	0.38	93
500	13.3	50	0.10	7.5
cw	–	55	–	–

More Q-switching regimes

- Slow, active Q-switch
  - Opening time is finite, include t-dependent loss during build-up
- Cavity dumping
  - Build-up to saturation, no output, then switch all out
- Passive Q-switch, saturable absorber
  - Third equation to keep track of N_{SA}
  - Gain builds to reach a higher N_{th}, then SA saturates during build up
  - Pulse energy, duration depend on pumping level
  - For CW pump, rep rate also depends on pumping level
  - Design: OD of unsaturated loss, beam size in SA

Mode-locking

- Q-switched pulse duration is limited by cavity photon lifetime. Cavity dump: round trip time
- For shorter pulses, we need broad spectral bandwidth
  - Run CW on many longitudinal modes
  - Random phase is just a noisy laser
  - Must lock phase of the longitudinal modes
- Active mode-locking
  - Intracavity device to modulate loss, synced up with the repetition rate
- Passive mode-locking
  - Nonlinear effect that leads to lower loss during pulse

Fourier transforms: t-ω domain

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{+i\omega t}\,dt = FT\{f(t)\}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}\,dt = FT^{-1}\{F(\omega)\}$$

- In EM, our signals are complex fields
- 1/2π factor is lumped into inverse transform
- ω is our frequency variable, not ν. This affects the normalization constants.
- Note signs of exponents: this is tied to our exp(-i ω t) convention
- Techniques
  - Analytic: apply transform IDs and theorems to decompose a transform into its parts
  - Analytic in Mathematica: can do some FTs but not always expressed in recognizable way
  - Graphical: after identifying components of a transform, sketch the anticipated result
  - Numerical: FFT for calculating complicated or realistic cases for modeling/data analysis

FT of a Gaussian pulse

- Starting integral: $\int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi}$
  - True even if z is complex

$$f(t) = e^{-t^2/t_0^2} \qquad FT\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} e^{-t^2/t_0^2} e^{+i\omega t}\, dt$$

- Complete the square in the exponent…

FT of a Gaussian is a Gaussian

- Starting integral: $\int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi}$
  - True even if z is complex

$$f(t) = e^{-t^2/t_0{}^2} \qquad FT\left\{f(t)\right\} = F(\omega) = \int_{-\infty}^{\infty} e^{-t^2/t_0{}^2} e^{+i\omega t}\, dt$$

- Complete the square in the exponent

$$-\frac{t^2}{t_0^2} + i\omega t = -\frac{1}{t_0^2}\left(t^2 - i\,\omega\, t\, t_0^2\right) = -\frac{1}{t_0^2}\left(\left(t - \frac{i}{2}\omega\, t_0^2\right)^2 + \frac{1}{4}\omega^2\, t_0^4\right)$$

$$= -\frac{1}{t_0^2}\left(t - \frac{i}{2}\omega\, t_0^2\right)^2 - \frac{1}{4}\omega^2\, t_0^2$$

  - Change variables: $z = \frac{1}{t_0}\left(t - \frac{i}{2}\omega\, t_0^2\right)$

$$F(\omega) = \int_{-\infty}^{\infty} e^{-t^2/t_0^2} e^{+i\omega t}\, dt = t_0 e^{-\frac{1}{4}\omega^2\, t_0^2} \int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi}\; t_0\; e^{-\frac{1}{4}\omega^2\, t_0^2}$$

Other transform pairs: FT{rect]=sinc and Dirac delta

- Rect(t/t0) $rect\left(\frac{t}{t_0}\right)=1 \text{ for } |t|<\frac{t_0}{2}$

$$F(\omega)=\int_{-\infty}^{\infty} rect(t/t_0)e^{+i\omega t}\,dt=\int_{-t_0/2}^{t_0/2} e^{+i\omega t}\,dt=\frac{1}{i\omega}\left(e^{+i\omega t_0/2}-e^{-i\omega t_0/2}\right)$$

$$=t_0\frac{\sin(\omega t_0/2)}{\omega t_0/2}=t_0\,\mathrm{sinc}(\omega t_0/2)$$

- Dirac delta $\int_{-\infty}^{\infty}\delta(t)dt=1$
  - Limit:

$$\delta(\omega)=\lim_{t_0\to\infty}FT\left\{\mathrm{rect}(t/t_0)\right\}=\lim_{t_0\to\infty}\left[t_0\,\mathrm{sinc}(\omega t_0/2)\right]$$

  - At ω=0, limit is ∞
  - ω≠0, limit is 0 in sense that integral over rapid osc sin( ) is 0
  - Normalization:

$$FT\{1\}=2\pi\delta(\omega) \qquad FT^{-1}\{1\}=\delta(t)$$

Time-bandwidth product

- "uncertainty principle" comes from FT relations

$$FT\left(e^{-t^2/t_0^2}\right) \rightarrow t_0\ e^{-\frac{1}{4}\omega^2\ t_0^2}$$

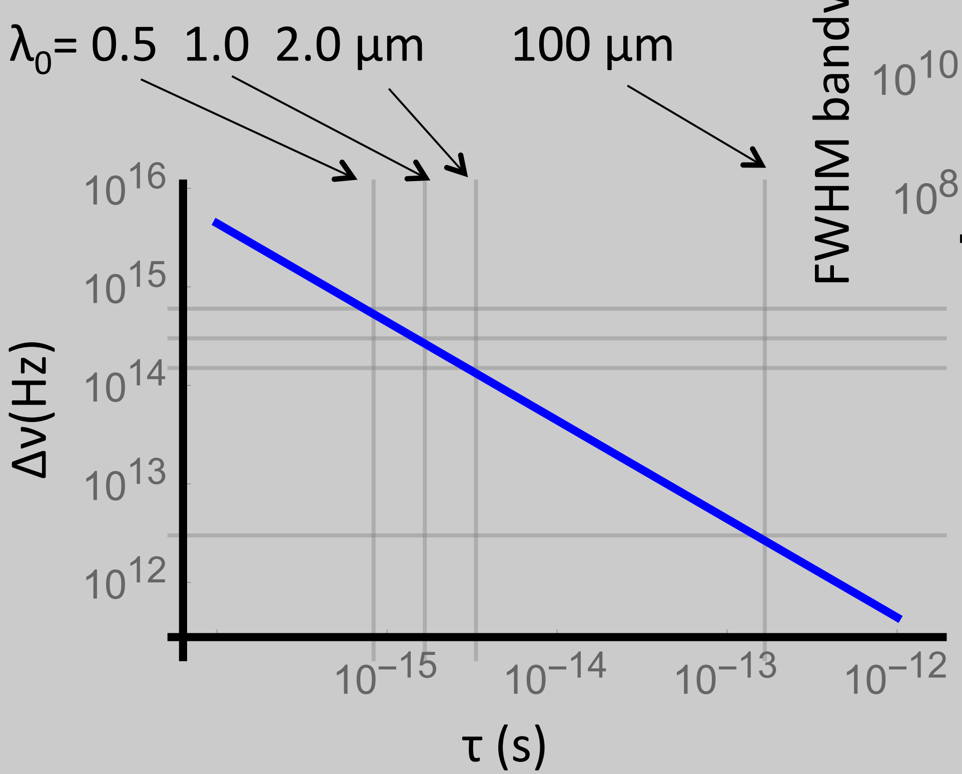
  - Pulse duration: t_0
  - Spectral width (bandwidth): $\delta\omega = 2/t_0$
  - Time-bandwidth product: $t_0\delta\omega = 2$
- This relation depends on how widths are defined
  - Here we've been using 1/e half width in the field
  - For FWHM in intensity: $E(t) = E_0 e^{-2\ln 2\, t^2/\tau^2} \rightarrow I(t) \propto e^{-4\ln 2\, t^2/\tau^2}$

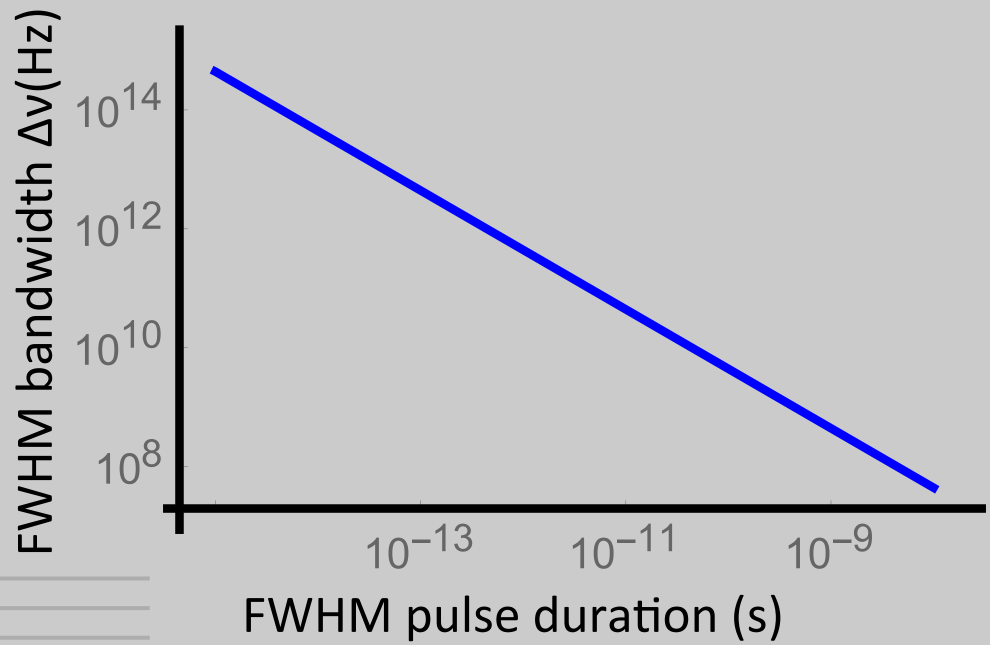
$$\tau = t_0\sqrt{2\ln 2} \qquad \Delta\omega = \delta\omega\sqrt{2\ln 2}$$

$$\boxed{t_0\delta\omega = 2 = \frac{\tau\Delta\omega}{2\ln 2} \rightarrow \tau\Delta\omega = 4\ln 2 \approx 2.77} \qquad \boxed{\tau\ \Delta\nu = \frac{4\ln 2}{2\pi} \approx 0.44}$$

Bandwidth for transform-limited pulses

- The bandwidth in frequency space is independent of the central frequency

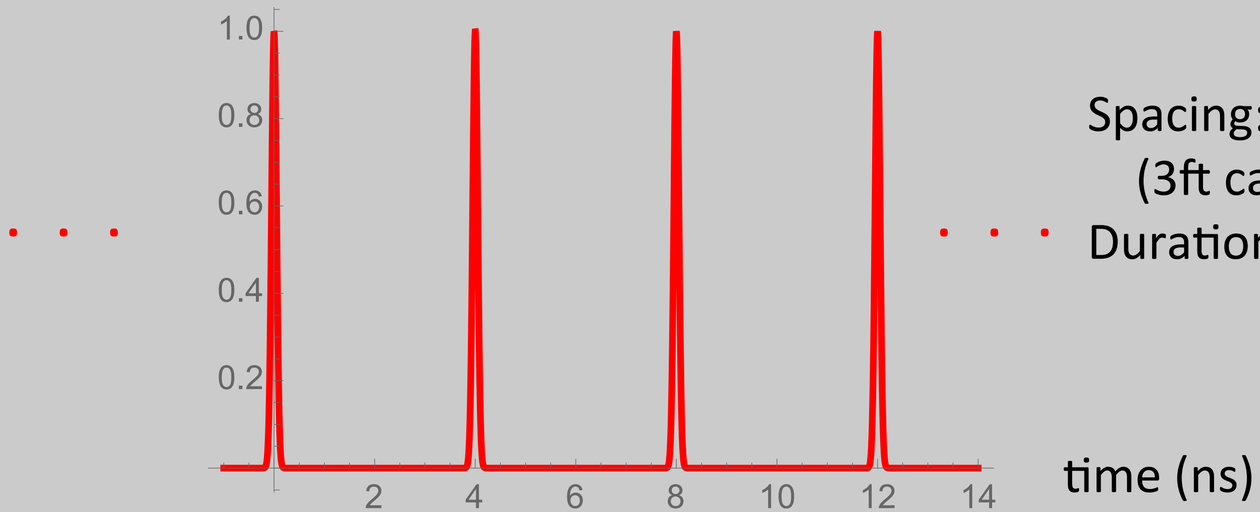




The shortest pulse is limited to ½ of an optical cycle

Train of Gaussian pulses

- Typical scenario: 1 pulse per round trip



Spacing: 6ns
(3ft cavity length)
Duration: 100ps

- What is spectrum of a pulse train?

Spectrum of a pulse pair

- Spectrum = |FT{field in time domain} |^2
  - Add a pulse to a copy of the pulse with time delay
  - Calculate the spectrum
- Spectrum of delayed pulse:

$$FT\{f(t-T)\} = \int_{-\infty}^{\infty} f(t-T)e^{+i\omega t}\,dt$$

Let t'= t-T

$$\int_{-\infty}^{\infty} f(t')e^{+i\omega(t'+T)}\,dt = F(\omega)e^{i\omega T}$$

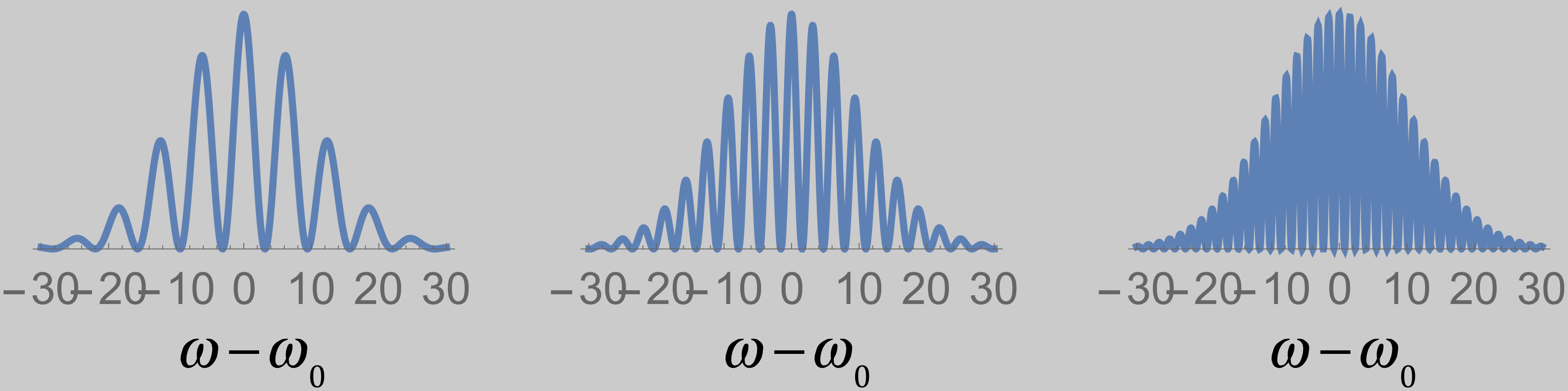
  - Give a phase shift (shift theorem) $FT\{E(t-T)\} = E(\omega)e^{i\omega T}$
- Spectrum of two pulses:

$$\left|E(\omega)+E(\omega)e^{i\omega T}\right|^2 = \left|E(\omega)\right|^2\left|1+e^{i\omega T}\right|^2 = \left|E(\omega)\right|^2 4\cos^2(\omega T/2)$$

Two pulse spectrum

- Spectral interference of two pulses is like the double-slit interference

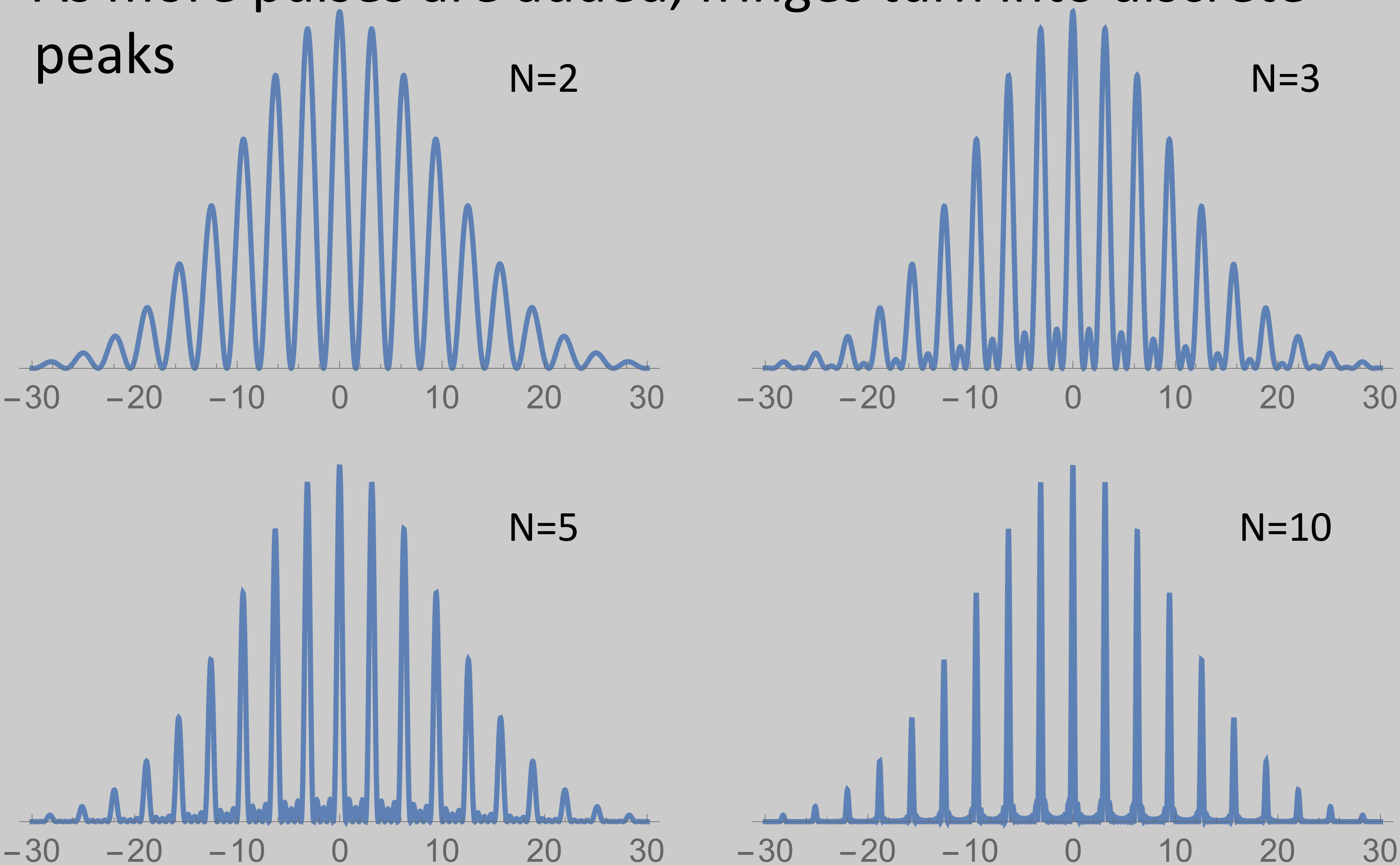
$$\left|E(\omega)+E(\omega)e^{i\omega T}\right|^2 = 4\left|E(\omega)\right|^2 \cos^2(\omega T/2)$$



Increasing delay ➔

Multipulse spectrum

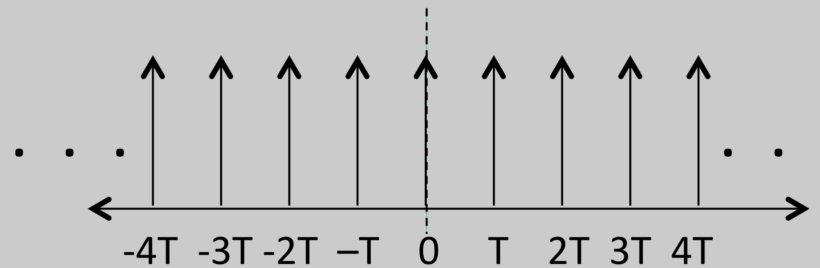
- As more pulses are added, fringes turn into discrete peaks



Comb function

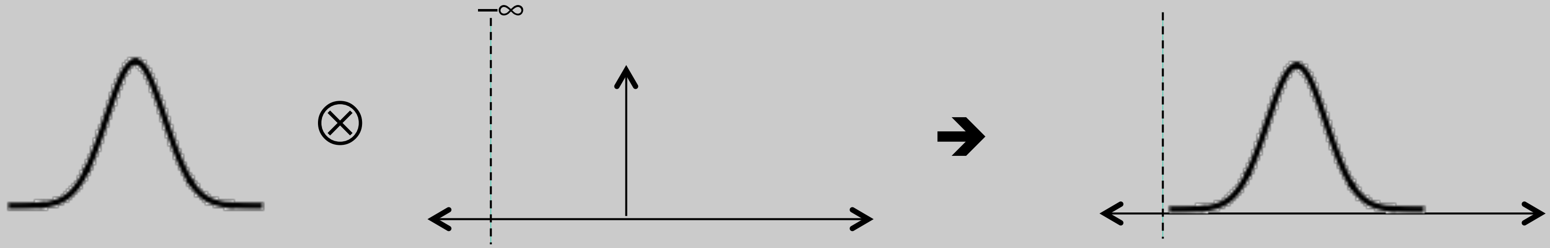
- Define the comb function

$$\mathrm{comb}\left(t/T\right) \equiv \sum_{n=-\infty}^{\infty} \delta\left(t - nT\right)$$

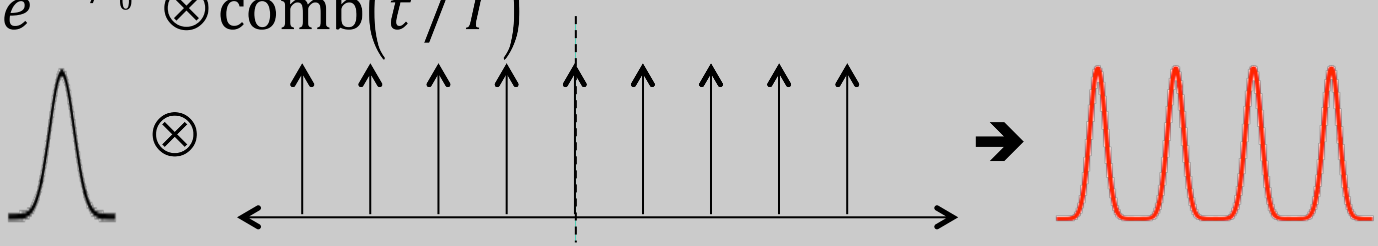


- A pulse train can be written as a convolution

$$f\left(t\right) \otimes \delta\left(t-T\right) = \int_{-\infty}^{\infty} f\left(t-t'\right)\delta\left(t'-T\right)dt' = f\left(t-T\right)$$



$$e^{-t^2/t_0^{\,2}} \otimes \mathrm{comb}\left(t/T\right)$$



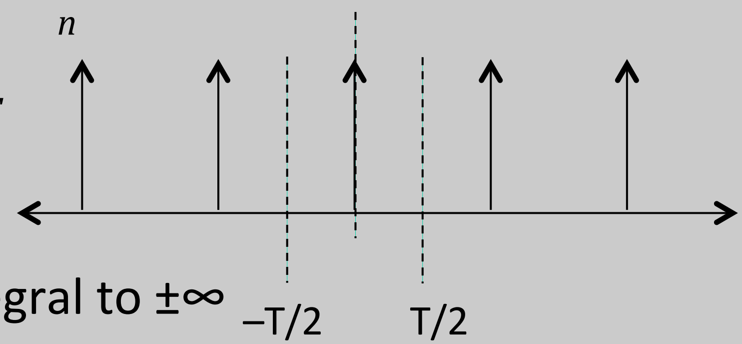
Array theorem: FT of comb( )

- Basic FT is straightforward:

$$f(t) = \text{comb}(t/T) \equiv \sum_{n=-\infty}^{\infty} \delta(t - nT) \qquad F(\omega) = \sum_{n=-\infty}^{\infty} FT\{\delta(t-nT)\} = \sum_{n=-\infty}^{\infty} e^{i\omega nT}$$

- This is actually a comb function also
- Since comb( ) is a periodic function (period T), we can write as a Fourier series: $f(t) = \sum_n c_n e^{i2\pi nt/T}$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-i2\pi nt/T} = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-i2\pi nt/T}$$



Integrate over one period, but we can extend integral to ±∞

$$c_n = \frac{1}{T} \qquad \therefore \text{comb}(t/T) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{2\pi nt/T}$$

Array theorem (cont)

- Now take FT:

$$f(t) = \text{comb}(t/T) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{2\pi nt/T}$$

$$F(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} FT\left\{e^{2\pi nt/T}\right\} = \frac{1}{T}\sum_{n=-\infty}^{\infty} 2\pi\delta\left(\omega + \frac{2\pi n}{T}\right)$$

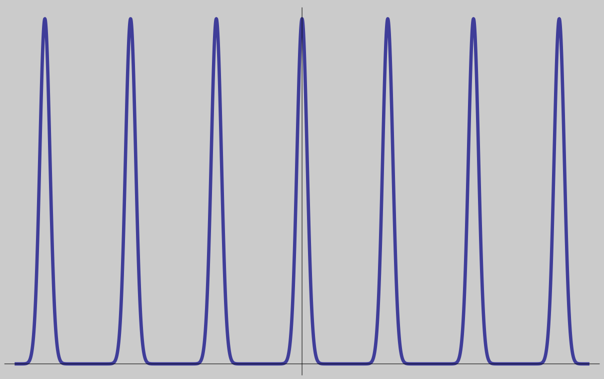
$$F(\omega) = \frac{2\pi}{T} comb\left(\frac{\omega}{2\pi/T}\right)$$

- So FT{comb} = comb
  - Frequency spacing $\Delta\omega=2\pi/T$ or $\Delta\nu=1/T$

Spectrum of a pulse train

- Gain envelope on longitudinal mode spectrum

$$e^{-t^2/t_0^{\,2}} \otimes \mathrm{comb}\left(t/T\right)$$



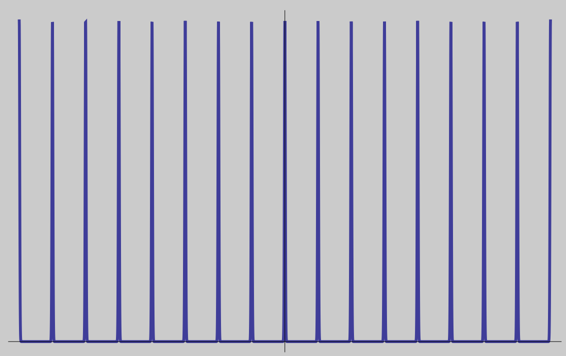
$$FT\left\{e^{-t^2/t_0^{\,2}} \otimes \mathrm{comb}\left(t/T\right)\right\}$$
$$= e^{-t_0^{\,2}\omega^2/4}\,\mathrm{comb}\left(\omega/\Delta\omega\right)$$

T is pulse spacing = round trip time in laser resonator

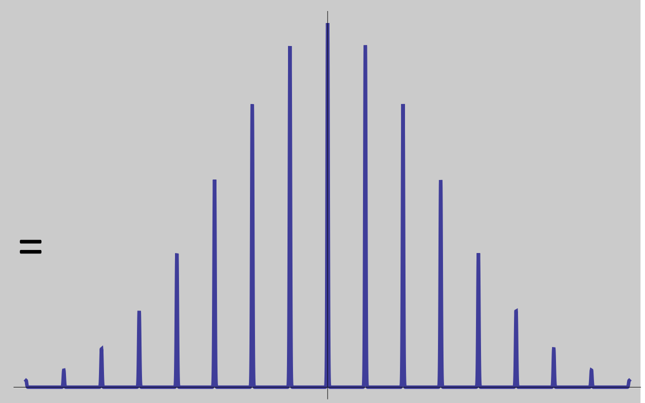
$\Delta\nu = 1/T$ = spacing of peaks in frequency

= longitudinal mode spectrum

X



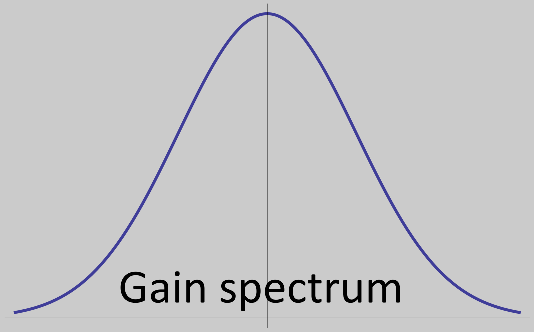
=



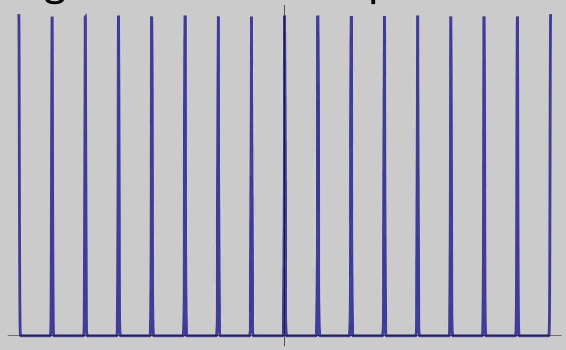
Spectrum of a pulse train

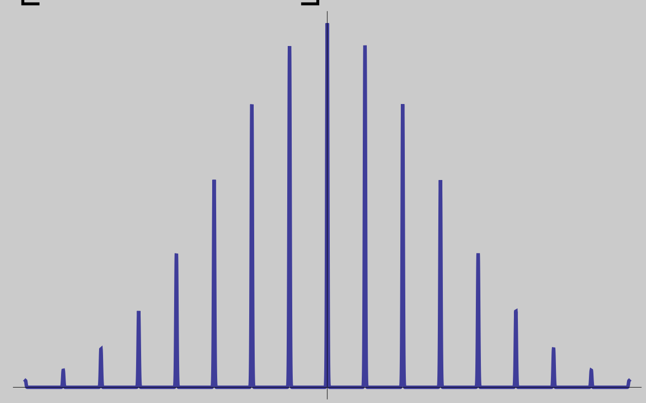
- Reverse reasoning: multiply gain envelope on longitudinal mode spectrum

$$\exp\left[-\frac{(\omega-\omega_0)^2}{\Delta\omega^2}\right] \qquad \mathrm{comb}(\omega/\delta\omega) \qquad \exp\left[-\frac{(\omega-\omega_0)^2}{\Delta\omega^2}\right]\cdot\mathrm{comb}(\omega/\delta\omega)$$



Longitudinal mode spectrum

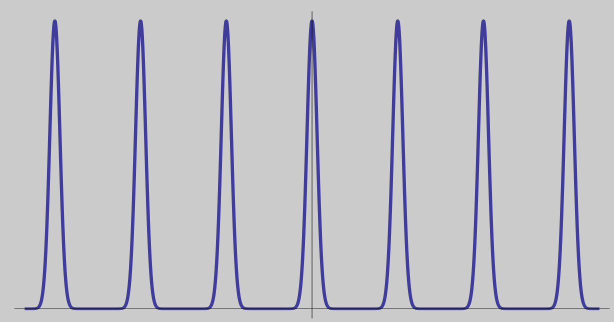




$$FT\left\{\exp\left[-\frac{(\omega-\omega_0)^2}{\Delta\omega^2}\right]\cdot\mathrm{comb}(\omega/\delta\omega)\right\}$$

$$= FT\left\{\exp\left[-\frac{(\omega-\omega_0)^2}{\Delta\omega^2}\right]\right\}\otimes\frac{1}{\delta\omega}\mathrm{comb}\left(t/(2\pi/\delta\omega)\right)$$

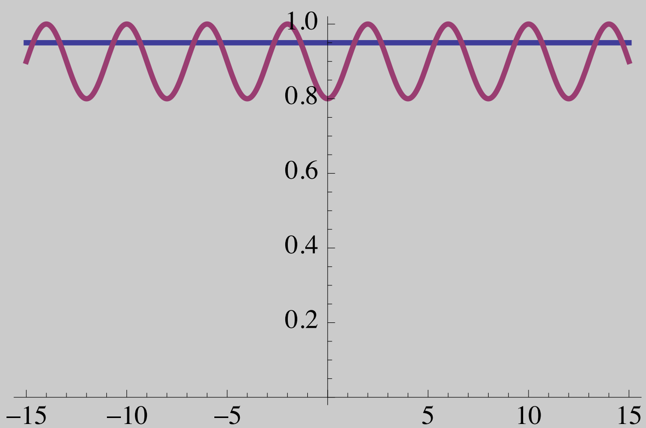
Pulse train output



Mode-locking: time-domain

- Active mode-locker: periodically modulate losses at RT time

$$f_{\mathrm{mod}}(t) = 1 - a\cos^2\left(2\pi t / T_{RT}\right)$$



Modulator acts as a "window" for a pulse to be transmitted

- Pulse duration is connected to modulation depth
- Picture doesn't explain how pulse can be much shorter than round trip time

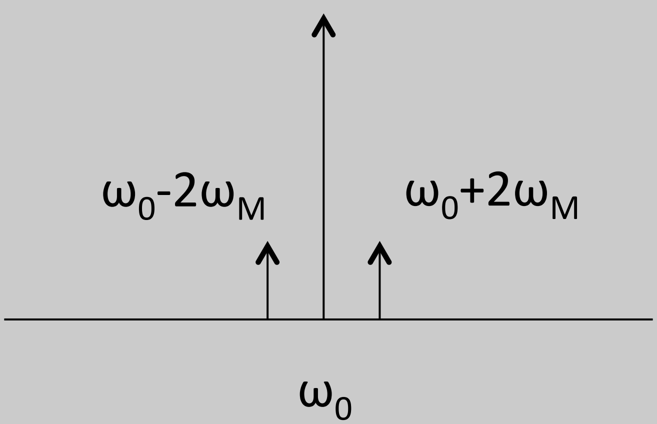
Mode locking: frequency domain

- Random phases produces noisy output
- Modes must be "locked" in phase to produce pulses
- Frequency domain representation of modulator:

$$f_{\text{mod}}(t) = 1 - a\cos^2(\omega_M t)$$

$$FT\left\{f_{\text{mod}}(t)\right\} = FT\{1\} - a\,FT\left\{\cos^2(\omega_M t)\right\}$$

$$\cos^2(\pi t / T_{RT}) = \frac{1}{4}\left(e^{i\omega_M t} + e^{-i\omega_M t}\right)^2 = \frac{1}{4}\left(2 + e^{i2\omega_M t} + e^{-i2\omega_M t}\right)$$



Adjust $2\omega_M$ to match longitudinal mode spacing.

Coherent seeding cascades to all modes with gain.

Active mode-locking

- Two mechanisms affect the evolution of the pulse
  - Time-dependent transmission narrows the pulse in time
  - ω-dependent gain narrows the spectrum (longer in time)
- Look for a pulse shape that repeats itself on each round trip (like spatial mode does)
- With small change to pulse for each individual element in cavity, we can treat them as distributed
  - Leads to a master differential equation for pulse
  - Solution for the pulse shape is an eigenfunction of the equation
- Assume $\tau_{21} >> T_{RT}$
- Assume laser is operating steady-state CW: one pulse per RT, e.g. 100 GHz repetition rate for a 5' long resonator.

Propagation equation: spectral gain

- Pulse sees saturated gain based on average intensity $\langle I \rangle$

$$g_0 = \frac{g}{1 + \langle I \rangle / I_{sat}} \qquad I_{sat} = \frac{h\nu_0}{\sigma_{pk}\tau_{21}} \qquad g = \sigma_{pk} N_0$$

Steady state small-sig gain coeff. $N_0 = N_{th}$

- Gain varies with frequency:

$$g_0(\omega) = \frac{g_0}{1 + \left(\frac{2(\omega - \omega_0)}{\Delta\omega_0} \right)^2}$$

Lorentzian profile

  - Expand this near the central frequency

$$g_0(\omega) \approx g_0 \left(1 - 4 \frac{(\omega - \omega_0)^2}{\Delta\omega_0^{\,2}} \right) \qquad G(\omega) = e^{g_0(\omega)}$$

Gain effect in frequency domain

- We want to develop propagation equation for the envelope of the pulse in the time domain.

$$E(t) = A(t)e^{-i\omega_0 t+\phi}$$

$$\rightarrow A(\omega-\omega_0) = FT\left\{A(t)e^{-i\omega_0 t}\right\} \equiv \int_{-\infty}^{\infty} A(t)e^{i(\omega-\omega_0)t}\,dt$$ Note use of shift thm

- In spectral domain, effect of gain is multiplicative:

$$A'(\omega-\omega_0) = A(\omega-\omega_0)G(\omega-\omega_0) = A(\omega-\omega_0)\exp\left[g_0\left(1-4\frac{(\omega-\omega_0)^2}{\Delta\omega_0{}^2}\right)\right]$$

- For small gain,

$$A'(\omega-\omega_0) \approx \left(1+g_0\left(1-4\frac{(\omega-\omega_0)^2}{\Delta\omega_0{}^2}\right)\right)A(\omega-\omega_0)$$

Gain effect in time domain

- Use inverse FT to calculate shape of amplified pulse

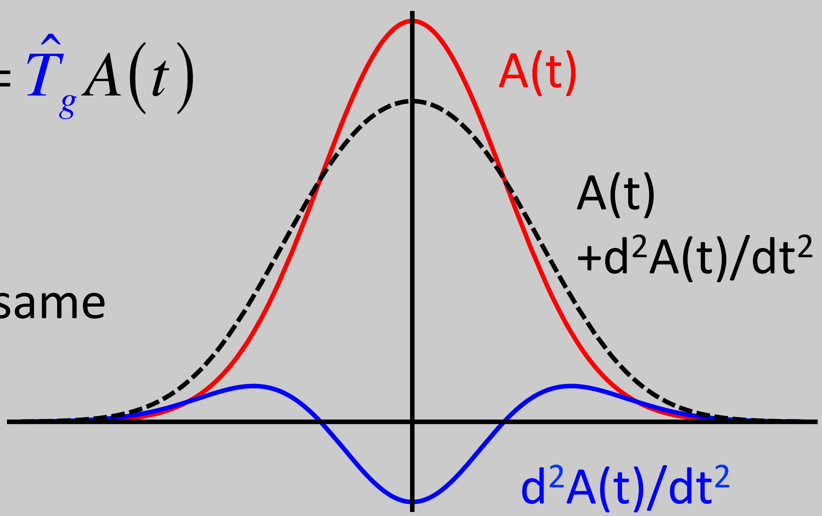
$$A'(t) = FT^{-1}\left\{A'(\omega - \omega_0)\right\}$$

- Make use of a FT property:

$$FT\left\{\frac{d^n}{dt^n}A(t)\right\} = \left[-i(\omega - \omega_0)\right]^n A(\omega - \omega_0)$$

Take FT definition, then derivative.

$$A'(t) = \left[1 + g_0\left(1 + \left(\frac{2}{\Delta\omega_0}\right)^2 \frac{d^2}{dt^2}\right)\right]A(t) = \hat{T}_g A(t)$$

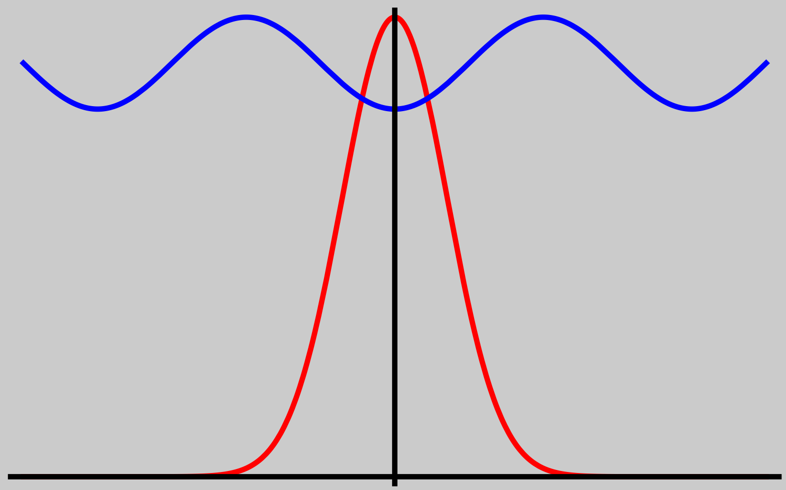


This operation in the time domain has the same effect as in the frequency domain.

Effect of modulator in time-domain

- Modulator loss shortens pulse in time domain
  - Double-pass transmission

$$T_m(t) = e^{-\gamma_m(1-\cos(\omega_m t))}$$



  - Assume low loss

$$T_m(t) \sim 1 - \gamma_m(1-\cos(\omega_m t))$$

  - Expand around pulse peak (t=0)

$$\hat{T}_m = 1 - \frac{\gamma_m}{2}(\omega_m t)^2$$

- Also write operator for passive loss:

$$\hat{T}_l = e^{-\gamma} \approx 1 - \gamma$$

Propagation equation for mode-locked pulse

- Effect of one round-trip on cavity is

$$A'(t) = \hat{T}_g \hat{T}_l \hat{T}_m A(t)$$

- Stable solution is an eigenfunction of this equation
- Differential form:

$$A'(t) = \left[g_0 \left(1 + \left(\frac{2}{\Delta\omega_0} \right)^2 \frac{d^2}{dt^2} \right) - \gamma - \frac{\gamma_m}{2} {\omega_m}^2 t^2 \right] A(t)$$

- Same form as Schrodinger eqn for SHO

Stable mode-locked pulse

- Solution follows Hermite-Gaussian form:

$$A(t) = H_n(\omega_p t) e^{-\omega_p^2 t^2/2} \qquad \omega_p = \left(\frac{\gamma_m}{2g_0}\right)^{1/4} \left(\frac{\omega_m \Delta\omega_0}{2}\right)^{1/2}$$

  - Only n = 0 is actually stable

- FWHM pulse duration:

$$\tau_p = \frac{2\sqrt{\ln 2}}{\omega_p} = 2\sqrt{\ln 2}\left(\frac{2g_0}{\gamma_m}\right)^{1/4}\left(\frac{2}{\omega_m \Delta\omega_0}\right)^{1/2} = \left(\frac{2\sqrt{2}\ln 2}{\pi^2}\right)^{1/2}\left(\frac{g_0}{\gamma_m}\right)^{1/4}\left(\frac{1}{\nu_m \Delta\nu_0}\right)^{1/2}$$

~0.45 ~1

- Nd:YAG example:

$$\Delta\nu_0 = 120\text{GHz} \qquad \nu_m = 76\text{MHz} \qquad \rightarrow \tau_p \approx 150\text{ps}$$

- Active mode-locking is limited by relatively slow action of the modulator (small ν_m)
  - Passive mode locking can produce much shorter pulses!

Passive mode locking

- Introduce a nonlinear effect so that there is lower loss with a pulse than without
  - If NL response is fast, the modulation can lead to much shorter pulses than with active mode-locking
- Kerr-lens mode-locking
  - Non-linear refractive index: higher refractive index during pulse
  - High refractive index in center of beam: positive lens
  - NL lens changes stability of cavity
  - Align cavity to be stable, lower loss with pulse than CW