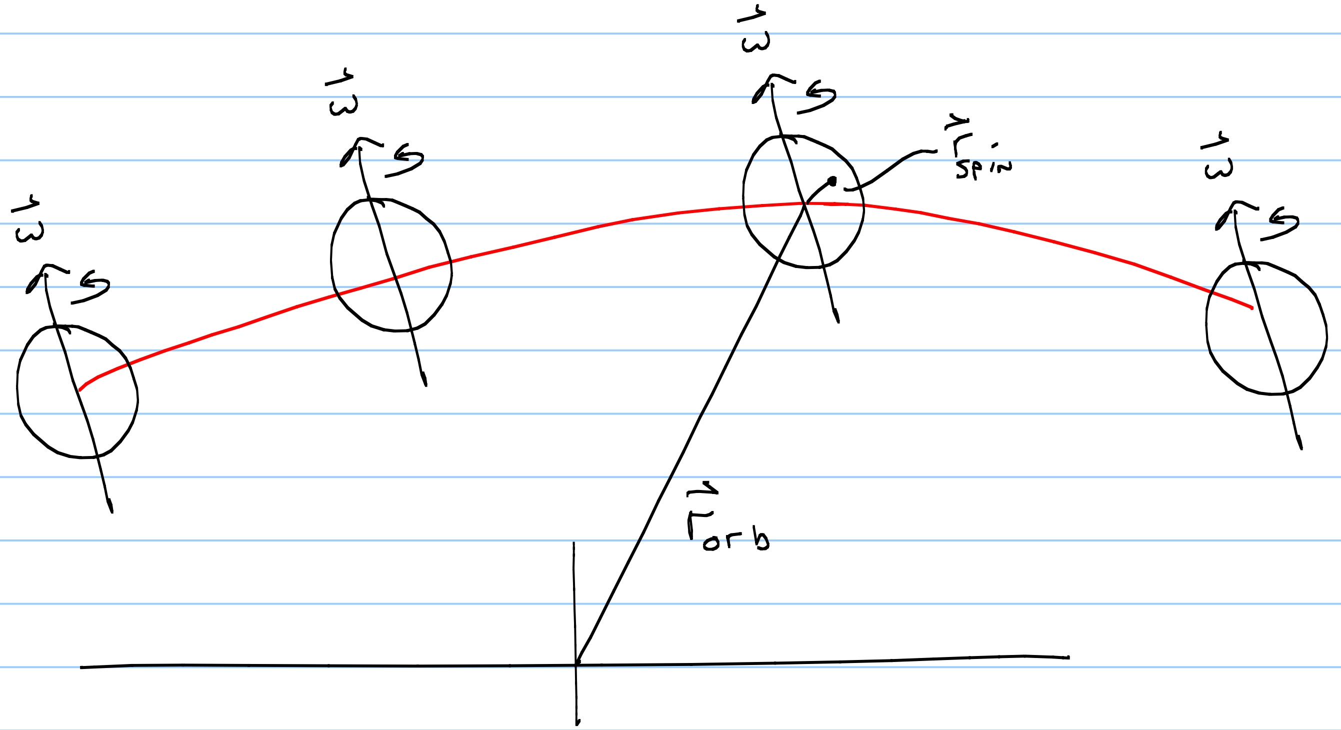
4-21-08

Note Title 4/21/2008

Classically $\vec{L} = \vec{r} \times \vec{p}$

$\vec{S} = I\vec{\omega}$

QM. elementary particles carry intrinsic spin angular momentum in addition to their orbital angular momentum



In QM, instead of considering the motion of a finite body we consider the motion of the wave-function. O'Hanian (Footnote Page 171) shows how this works starting with the classical EM field.

Algebraic theory of SAM (spin ang. momentum) identical to that of OAM (orb. ang momentum).

$\vec{S}$, $S_z$

$$S^2|s,m\rangle = \hbar^2 s(s+1)|s,m\rangle$$

$$S_z|s,m\rangle = \hbar\, m|s,m\rangle$$

and $$S_\pm|s,m\rangle = \hbar\sqrt{s(s+1)-m(m\pm1)}\;|s,m\pm1\rangle$$

cf. problem 4.18

$s = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$ $\qquad m = -s, -s+1, \cdots s-1, s$

similarly

$$\left.\begin{aligned}[S_x, S_y] &= i\hbar S_z\\ [S_y, S_z] &= i\hbar S_x\\ [S_z, S_x] &= i\hbar S_y\end{aligned}\right\}\ \text{cyclic permut.}$$

| Particle | $s$ |
|---|---|
| π-meson | 0 |
| electron | $\frac{1}{2}$ |
| photon | 1 |
| δ | $\frac{3}{2}$ |
| graviton | 2 |

} characterizes each particle

whereas OAM can take on any value $\ell$ and can be perturbed by experiment

# Spin $\frac{1}{2}$

(protons, neutrons, electrons, quarks)

$s = \frac{1}{2}$ so $m = -\frac{1}{2}, \frac{1}{2}$

$$|s, m\rangle \Rightarrow \underbrace{\begin{matrix} |\frac{1}{2}, \frac{1}{2}\rangle \\ |\frac{1}{2}, -\frac{1}{2}\rangle \end{matrix}}_{\text{2 spin eigenstates for a spin } \frac{1}{2} \text{ particle}}$$

So we can represent these as 2-component vectors

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \underline{\text{spinors}}$$

"spin up" "spin down"

$$S^2 \chi_+ = \hbar^2 \left(\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\right) \chi_+ = \tfrac{3}{4} \hbar^2 \chi_+$$

$$S^2 \chi_- = \hbar \left(\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\right) \chi_- = \tfrac{3}{4} \hbar^2 \chi_-$$

$$\boxed{s = \tfrac{1}{2}}$$

Let us represent $S^2$ in the basis defined by the spinors:

$$S^2 \equiv \begin{pmatrix} c & d \\ e & f \end{pmatrix}$$

so $\begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} c \\ e \end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$c = 3/4\hbar^2 \quad e = 0$

$\begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = $

$d = 0$

$f = 3/4\hbar^2$

$$S^2 = 3/4\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

on the other hand

$$S_z \chi_+ = \frac{\hbar}{2}\chi_+$$

$$S_z \chi_- = -\frac{\hbar}{2}\chi_-$$

so $\begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$S_+ \chi_- = \hbar \chi_+ \qquad S_- \chi_+ = \hbar \chi_-$

$S_+ \chi_+ = 0 \qquad S_- \chi_- = 0$

$\Rightarrow \quad S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\underbrace{\text{N.B. } S_+ + S_-}$ are not Hermitian

$$\begin{array}{l} S_+ = S_x + i S_y \\ S_- = S_x - i S_y \\ \hline \end{array}$$

$$\left.\begin{array}{l} S_+ + S_- = 2 S_x \\ S_+ - S_- = 2 i S_y \end{array}\right\} \quad \begin{array}{l} S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{array}$$

$$\left.\begin{array}{l} S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \frac{\hbar}{2}\sigma_x \\ S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \equiv \frac{\hbar}{2}\sigma_y \\ S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \frac{\hbar}{2}\sigma_z \end{array}\right\} \text{Pauli spin matrices}$$

# Spin msmt.

Let $\chi = a\chi_+ + b\chi_-$ be a generic spinor where

$$\chi_+ = \begin{pmatrix}1\\0\end{pmatrix} \qquad \chi_- = \begin{pmatrix}0\\1\end{pmatrix}$$

spin up spin down

$$S_z\chi_+ = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \underset{\text{eigenvalue}}{\frac{\hbar}{2}}\begin{pmatrix}1\\0\end{pmatrix} \leftarrow \text{eigenvector}$$

$$S_z\chi_- = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \underset{\text{eigenvalue}}{-\frac{\hbar}{2}}\begin{pmatrix}0\\1\end{pmatrix} \nwarrow \text{eigenvector}$$

$$S_z\chi = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}a\chi_+ + \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}b\chi_-$$

$$= a\frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix} - b\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= a\left(\frac{\hbar}{2}\right)\chi_+ + b\left(-\frac{\hbar}{2}\right)\chi_-$$

i.e. prob. of measuring $\frac{\hbar}{2}$ is $|a|^2$

prob. of measuring $-\frac{\hbar}{2}$ is $|b|^2$

So $|a|^2 + |b|^2 = 1$

A spinor $\chi = a\chi_+ + b\chi_-$ must Therefore be normalized.

Suppose we want to measure $S_x$?

What are the eigenvalues of $S_x$?

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Char. polynomial: $\text{Det}\begin{pmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{pmatrix} = 0$

$$\Rightarrow \quad \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0$$

$\lambda = \pm\frac{\hbar}{2}$ Same as $S_x$

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\Downarrow$ w/ eigenvalue $\frac{\hbar}{2}$ $\qquad$ $\Downarrow$ w/ eigenvalue $-\frac{\hbar}{2}$

e.g. $\chi = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$

$$\chi^2 = \frac{1}{6}\left((1+i)(1-i) + 4\right) = 1$$

If we measure $S_z$ what is the prob. of getting $\pm \hbar/2$ ?

$$\chi = \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$a = \frac{1}{\sqrt{6}}(1+i)$ $\quad$ $b = \frac{2}{\sqrt{6}})$

$|a|^2 = \frac{2}{6} = \frac{1}{3}$ $\quad$ $|b|^2 = \frac{4}{6} = \frac{2}{3}$

Prob. of measuring $+\frac{\hbar}{2}$ is $\frac{1}{3}$

$-\hbar/2$ $\quad$ $2/3$

For $S_x$ : $\frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = a \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(1+i) = a\sqrt{3} + b\sqrt{3}$

$2 = a\sqrt{3} - b\sqrt{3}$

$3 + i = 2\sqrt{3}\, a$ $\quad$ $a = \frac{3+i}{2\sqrt{3}}$

$-1 + i = 2\sqrt{3}\, b$ $\quad$ $b = \frac{-1+i}{2\sqrt{3}}$

$|a|^2 = \frac{10}{12} = \frac{5}{6}$ $\quad$ $|b|^2 = \frac{2}{12} = \frac{1}{6}$

Show that $\langle S_x \rangle = \frac{\hbar}{3}$