F.T. examples

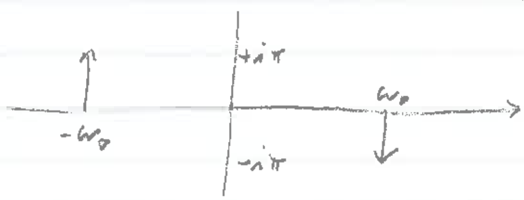
1) $\mathcal{F}\{\cos\omega_0 t\} = \frac{1}{2}\mathcal{F}\{e^{i\omega_0 t} + e^{-i\omega_0 t}\}$
(real, even)

$$= \frac{1}{2}\left(2\pi\,\delta(\omega-\omega_0) + 2\pi\,\delta(\omega+\omega_0)\right)$$
$$= \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$

real, even

2) $\mathcal{F}\{\sin\omega_0 t\} = \frac{1}{2i}\cdot 2\pi\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$
(real, odd)

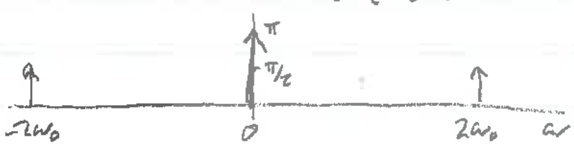
$$= \pi i\left(-\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)$$



imag, odd

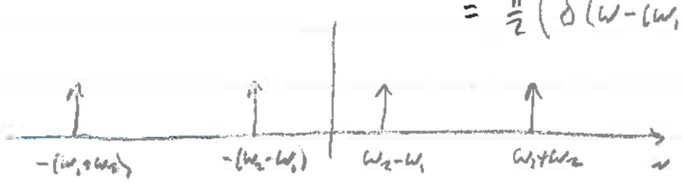
3) $\mathcal{F}\{\cos^2\omega_0 t\} = \mathcal{F}\{\frac{1}{4}(e^{2i\omega_0 t} + e^{-2i\omega_0 t} + 2)\}$

$$= \frac{\pi}{2}\left(\delta(\omega-2\omega_0) + \delta(\omega+2\omega_0) + 2\,\delta(\omega)\right)$$



4) $\mathcal{F}\{\cos\omega_1 t\,\cos\omega_2 t\} = \mathcal{F}\{\frac{1}{4}(e^{i(\omega_1+\omega_2)t} + e^{-i(\omega_1+\omega_2)t} + e^{i(\omega_1-\omega_2)t} + e^{-i(\omega_2-\omega_1)t})\}$
$\omega_2 > \omega_1$

$$= \frac{\pi}{2}\Big(\delta(\omega-(\omega_1+\omega_2)) + \delta(\omega+\omega_1+\omega_2) + \delta(\omega-(\omega_1-\omega_2)) + \delta(\omega+(\omega_1-\omega_2))\Big)$$



Parseval's theorem: energy is same in both domains

$$\int |f(t)|^2 dt = \frac{1}{2\pi}\int |F(\omega)|^2 d\omega \qquad \text{note Abs[ ]}^2$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int f(t)\, e^{i\omega t} dt\right] \underbrace{\left[\int f(t')\, e^{i\omega t'} dt'\right]^*}_{\int f^*(t')\, e^{-i\omega t'} dt'} d\omega$$

integrate on ω: $\frac{1}{2\pi}\int e^{i\omega(t-t')} d\omega = \mathfrak{F}^{-1}\{e^{i\omega t}\} = \delta(t'-t)$

$$\int |f(t)|^2 dt = \frac{1}{2\pi}\int dt\, f(t) \int dt'\, f^*(t')\, 2\pi\, \delta(t'-t) \quad \checkmark$$

Example: Gaussian pulses

$$F(\omega) = \sqrt{\pi}\, t_0\, e^{-\omega^2 t_0^2/4}$$

$$\frac{1}{2\pi}\int |F(\omega)|^2 d\omega = \frac{1}{2\pi}\pi t_0^2 \int e^{-\omega^2 t_0^2/2} d\omega = \frac{1}{2} t_0^2 \frac{\sqrt{2}}{t_0}\sqrt{\pi} = \sqrt{\frac{\pi}{2}}\, t_0$$

let $z = \omega t_0/\sqrt{2}$ $\quad dz = \frac{t_0}{\sqrt{2}} d\omega$

$$\int |f(t)|^2 dt = \int e^{-2t^2/t_0^2} dt = \sqrt{\pi}\frac{t_0}{\sqrt{2}} \quad \checkmark$$

$z = \sqrt{2}\, t/t_0, \quad dz = \frac{\sqrt{2}}{t_0} dt$

Parseval's theorem

FT gives a different *representation* of the signal. Energy must be conserved.

$$\int \left|f(t)\right|^2 dt = \frac{1}{2\pi}\int \left|F(\omega)\right|^2 d\omega$$

$$= \frac{1}{2\pi}\int \left[\int f(t)e^{i\omega t}\,dt\right]\left[\int f(t')e^{i\omega t'}\,dt'\right]^* d\omega$$

Note *independent* integrals for t, t'
Apply conjugation inside integral

$$= \frac{1}{2\pi}\int \left[\int f(t)e^{i\omega t}\,dt\right]\left[\int f^*(t')e^{-i\omega t'}\,dt'\right] d\omega$$

Gather ω terms

$$= \int dt\, f(t)\int dt'\, f^*(t')\left(\frac{1}{2\pi}\int e^{i\omega(t-t')}\,d\omega\right) == \int dt\, f(t)\int dt'\, f^*(t')\delta(t'-t)$$

$$= \frac{1}{2\pi}\int dt\, f(t)\int dt'\, f^*(t')2\pi\delta(t'-t) = \frac{1}{2\pi}\int dt\, f(t)f^*(t)$$

example: <u>square pulse</u>

envelope $\quad \text{rect}(t/t_0) = f_1(t)$

wave $\quad e^{-i\omega_0 t} = f_2(t)$

$$\left.\begin{matrix} \text{rect}(t/t_0) = f_1(t) \\ e^{-i\omega_0 t} = f_2(t) \end{matrix}\right\} f(t) = f_1(t) f_2(t)$$

<u>direct</u> $$F(\omega) = \int_{-t_0/2}^{t_0/2} e^{-i\omega_0 t} e^{i\omega t} \, dt$$

$$= \frac{1}{i(\omega-\omega_0)} \left(e^{i(\omega-\omega_0)t} - e^{-i(\omega-\omega_0)t} \right)$$

shift thm:

$$\mathcal{F}\{f(t-t_0)\} = e^{+i\omega t_0} F(\omega)$$

$$\mathcal{F}^{-1}\{F(\omega-\omega_0)\} = e^{-i\omega_0 t} f(t)$$

here $$\mathcal{F}\{\text{rect}(t/t_0) \, e^{-i\omega_0 t}\} = t_0 \, \text{sinc}\left(\frac{(\omega-\omega_0)}{2} t_0\right)$$

check
conservation of energy

$$\int |f(t)|^2 \, dt = \int_{-t_0/2}^{t_0/2} 1 \, dt = t_0$$

$$\int |F(\omega)|^2 \, d\omega = \int t_0^2 \, \text{sinc}^2\left(\frac{(\omega-\omega_0)t_0}{2}\right) d\omega$$

in general: $\qquad = 2\pi \, t_0$

Parseval's theorem:

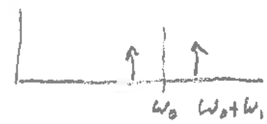
$$\int |f(t)|^2 \, dt = \frac{1}{2\pi} \int |F(\omega)|^2 \, d\omega$$

Example exercises

Amplitude modulator

$E(t)$ $\quad e^{-i\omega_0 t}$ in $\longrightarrow \boxed{A(t) = \cos \omega_1 t} \longrightarrow \cos \omega_1 t \, e^{-i\omega_0 t}$

$E(\omega)$



if $A(t) = \cos^2 \omega_1 t \longrightarrow$

Phase modulator

modulation on the refractive index $n(t) = n_0 (1 + A \cos \omega_1 t)$

if $A \ll 1$, what is output spectrum? (1st order)

$$E_{out}(t) = E_0 e^{-i\omega_0 t} e^{i k_0 z \cdot n(t)} = E_0 e^{i(kz - \omega_0 t)} e^{i k z A \cos \omega_1 t}$$

for $kz \cdot A \ll 1$ (small phase shifts)

$$E_{out}(t) \approx E_{out}(t, A=0)(1 + i k z A \cos \omega_1 t)$$

$$\tilde{E}_{out}(\omega) = \tilde{E}_{in}(\omega) + i \frac{kzA}{2}\left(\tilde{E}_{in}(\omega - \omega_1) + \tilde{E}_{in}(\omega + \omega_1)\right)$$

if $E_{in}(\omega) \sim E_0 \, \delta(\omega - \omega_0) \longrightarrow$

cross terms in $|E|^2$ are zero.

to second order $E_{out}(t) = E_{out}(A=0)(1 + i k z A \cos \omega_1 t - (kzA)^2 \cos^2 \omega_1 t)$

$\longrightarrow$

Convolution theorem:

consider $\mathcal{F}\{f(t)g(t)\} = \int f(t)g(t)\,e^{i\omega t}\,dt$

substitute: $g(t) \to G(\omega)$

$$\to \int dt\, f(t)\, e^{i\omega t}\, \frac{1}{2\pi}\int G(\omega')\, e^{-i\omega' t}\, d\omega'$$

ω' is diff't from ω

Swap integration order:

$$\frac{1}{2\pi}\int G(\omega')\,d\omega' \underbrace{\int f(t)\, e^{i(\omega-\omega')t}\,dt}_{F(\omega-\omega')}$$

$$\therefore \underbrace{\mathcal{F}\{f(t)g(t)\}}_{\text{product}} = \underbrace{\frac{1}{2\pi}\int G(\omega')\,F(\omega-\omega')\,d\omega'}_{\text{convolution}} = \frac{1}{2\pi}F(\omega)\otimes G(\omega)$$

product $\xrightarrow{\mathcal{F}}$ convolution

Corresponding in ω-space:

$$\mathcal{F}^{-1}\{F(\omega)\,G(\omega)\} = f(t)\otimes g(t)$$

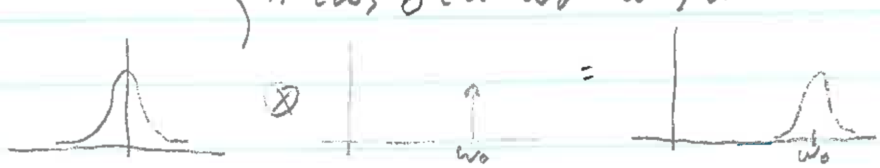
Convolution integral - see web link. (Erik Cheever / Swarthmore)

- easy to do with δ-functions.

Example:

$$\mathcal{F}\{f(t)\,e^{-i\omega_0 t}\} = \frac{1}{2\pi}F(\omega)\otimes\left(2\pi\,\delta(\omega-\omega_0)\right)$$

$$= \int F(\omega')\,\delta(\omega-\omega_0-\omega')\,d\omega' = F(\omega-\omega_0)$$



Convolution theorem

FT of the product of two functions is the convolution of the transforms

$$FT\left\{f(t)g(t)\right\} = \frac{1}{2\pi}F(\omega)\otimes G(\omega)$$

$$FT\left\{f(t)g(t)\right\} = \int f(t)g(t)e^{i\omega t}\,dt$$

$$= \int f(t)\left[\frac{1}{2\pi}\int G(\omega')e^{-i\omega' t}\,d\omega'\right]e^{i\omega t}\,dt$$

Note *independent* variables for ω, ω'

$$= \frac{1}{2\pi}\int G(\omega')d\omega'\int f(t)e^{i(\omega-\omega')t}\,dt$$

Swap order of integration: t first

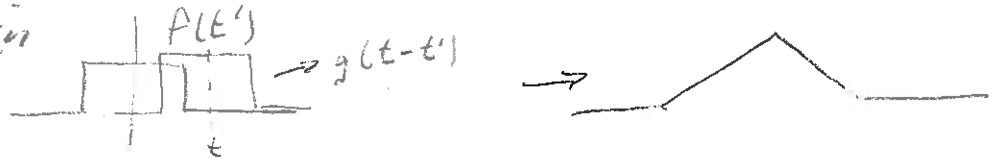
$$= \frac{1}{2\pi}\int F(\omega-\omega')G(\omega')d\omega' = \frac{1}{2\pi}F(\omega)\otimes G(\omega)$$

Inverse FT of the product of two functions is the convolution of the transforms

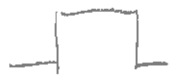
$$FT^{-1}\left\{F(\omega)G(\omega)\right\} = f(t)\otimes g(t)$$

$$h(\tau) = f(t) \otimes g(t) \equiv \int f(t)\, g(\tau - t)\, dt$$

<u>Convolution</u>



each pt of t is integral of product of shifted.

$f(t) =$  $g(t) =$ 

$f(t')\, \underline{g(t-t')}$ → 

↓

flips fcn backwards

Correllation

$$h_c(t) \equiv \int f(t')\, g^*(t+t')\, dt'$$

no flip

correllation fcn at $t=0$ measures how similar functions are.

Autocorrellation

$$h_{AC}(t) = \int f(t')\, f^*(t+t')\, dt'$$

transforms:

first note $h_c = -\int f(-\tau)\, g^*(t-\tau)\, d\tau$

$$= -f(-t) \otimes g^*(t)$$

$$H_c = \mathcal{F}\{ h_c(t) \}$$