Propagation of non monochromatic light.

single frequency:

$$E(t) = E_0 e^{-i\omega_0 t}$$

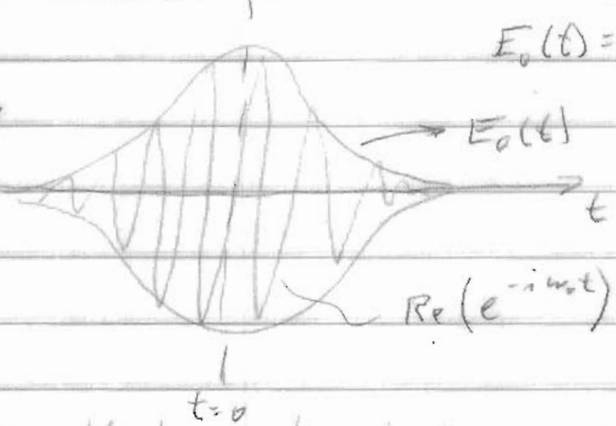
quasi monochromatic:

$$E(t) = E_0(t) e^{-i\omega_0 t}$$

ω_0 = carrier freq.

$E_0(t)$ = envelope.

ex. Gaussian



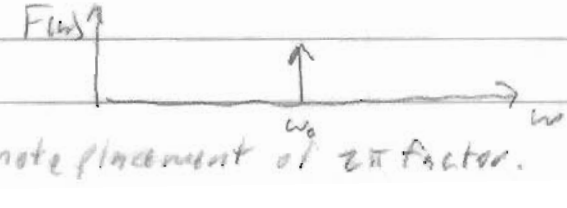
Any modulation in the time domain requires a mixture of frequencies. Use Fourier transforms to get spectrum:

Define $\mathcal{F}\{f(t)\} \equiv F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega$$

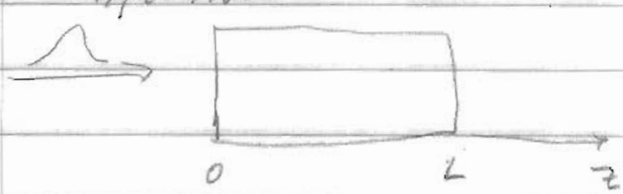
- note opposite signs in $\mathcal{F}\{\ \}$ and $\mathcal{F}^{-1}\{\ \}$
- choice in signs is consistent with $e^{-i\omega_0 t}$ convention

$$\mathcal{F}\{e^{-i\omega_0 t}\} = \int_{-\infty}^{\infty} e^{i(\omega-\omega_0)t} dt \equiv 2\pi\delta(\omega-\omega_0)$$



- note placement of 2π factor.

Dispersion



by travelling from $z=0$ to $z=L$,

$$E(z=0)\, e^{ik_0 nL}$$

since $n = n(\omega)$ we must apply this in the freq. domain

$$\tilde{E}_{out}(\omega) = \tilde{E}_{in}(\omega)\, e^{i\frac{\omega}{c} n(\omega) L}$$

note that if $n(\omega)$ is real,

$|E_{out}(\omega)|^2$ is unchanged by medium

i.e. no absorption

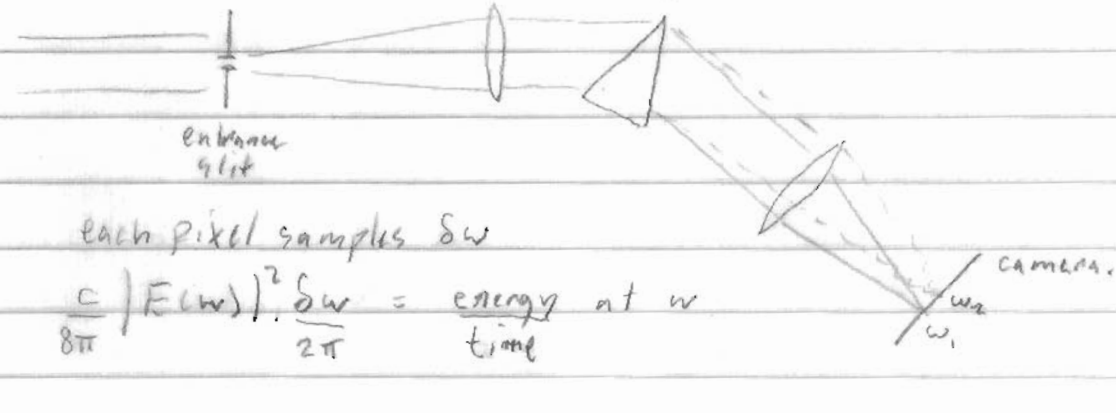
$\frac{\omega}{c} n(\omega) L$ is just a phase shift, depends on ω.

$\equiv \Phi(\omega)$ spectral phase.

In a linear system, each frequency component propagates on its own.

$|E_{out}(\omega)|^2 \propto$ spectral intensity

this is what is measured by a spectrometer:



each pixel samples $\delta\omega$

$$\frac{c}{8\pi} |E(\omega)|^2 \cdot \frac{\delta\omega}{2\pi} = \frac{\text{energy}}{\text{time}} \text{ at } \omega$$

Gaussian pulse propagating through a dispersive medium

let $E_{in}(t) = E_0 a(t) e^{-i\omega_0 t}$

with $a(t) = \exp(-t^2/\tau^2)$

here no extra phases

calc. spectrum: $A(\omega) = \mathcal{F}\{a(t)e^{-i\omega_0 t}\}$

1) $$\mathcal{F}\{e^{-t^2/\tau^2}\} = \int_{-\infty}^{\infty} e^{-t^2/\tau^2 + i\omega t}\, dt$$

complete square in exponent to get to form:

$$\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$$

$$-\frac{1}{\tau^2}(t^2 - i\omega\tau^2 t) = -\frac{1}{\tau^2}\left[\left(t - \frac{i\omega\tau^2}{2}\right)^2 + \frac{\omega^2\tau^4}{4}\right]$$

$$\rightarrow e^{-\frac{\omega^2\tau^2}{4}} \int_{-\infty}^{\infty} e^{-\frac{(\;)}{\tau^2}} dt$$

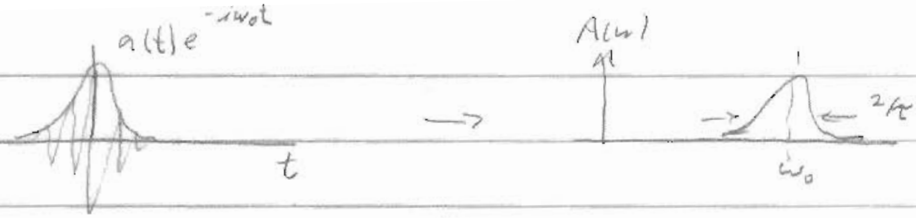
let $u = \left(t - \frac{i\omega\tau}{2}\right)/\tau \qquad du = \frac{1}{\tau} dt$

$$\mathcal{F}\{e^{-t^2/\tau^2}\} = \tau \cdot \sqrt{\pi}\, e^{-\omega^2\tau^2/4}$$ another Gaussian.

2) use shift theorem:

$$\int f(t) e^{-i\omega_0 t} e^{i\omega t} dt = F(\omega - \omega_0)$$

$$\therefore A(\omega) = \sqrt{\pi\tau^2}\, e^{-(\omega-\omega_0)^2\tau^2/4}$$



in t: $1/e$ half width $= \tau$

in ω: " $= 2/\tau$

$\Delta t \, \Delta\omega = 2$ uncertainty principle

QM: photon energy $= \hbar\omega$

$\Delta E \, \Delta t = 2\hbar$ note QM definition of width is different.

photon counting: $\hbar\omega$ can be anywhere w/in range $\Delta\omega$

individual photons share same wavefunction as packet

$E \sim \Psi$

$|E|^2 \sim |\Psi|^2$ energy density ~ probability density

$E(t)$ and $\tilde{E}(\omega)$ are different <u>representations</u> of same signal

$\therefore$ expect the same total energy:

$$\int |E(t)|^2 dt = \underbrace{\frac{1}{2\pi}}_{\text{comes in from proof}} \int |E(\omega)|^2 d\omega$$

this is Parseval's thm.

check:
$$\int E_0^2 e^{-2t^2/\tau^2} dt = E_0^2 \frac{\tau}{\sqrt{2}} \int e^{-u^2} du = E_0^2 \tau \sqrt{\frac{\pi}{2}}$$

$$\frac{1}{2\pi} \int E_0^2 \pi \tau^2 e^{-(\omega-\omega_0)^2 \tau^2/2} d\omega = \frac{\tau^2}{2} E_0^2 \sqrt{\frac{2}{\tau^2}} \sqrt{\pi} \quad \checkmark$$

$\frac{c}{4\pi} |E(t)|^2 =$ intensity (time dependent) $= I(t)$ e.g. W/cm^2

$\int I(t) dt =$ energy fluence e.g. J/cm^2

Origin of group velocity

we know that phase velocity is $V_{ph} = \omega/k$

group velocity is $V_g = d\omega/dk$.

expect that the center of a pulse envelope travels at V_g. $\rightarrow$ $E_0(t) \rightarrow E_0(t - z/V_g)$

Note that t is relative to the pulse peak:

$e^{-t^2/\tau^2} \rightarrow e^{-(t-z/V_g)^2/\tau^2}$ moving peak.

Derive group velocity:

after propagation by L

$$E_{out}(\omega) = \underbrace{E_0 \tau \sqrt{\pi}\, e^{-\tau^2(\omega-\omega_0)^2/4}}_{\text{input pulse spectrum}} \underbrace{e^{i\omega n(\omega) L/c}}_{\text{propagation effect}}$$

To describe pulse in the time domain, we must transform back.

But $e^{i\omega n L/c}$ is complicated

$\therefore$ approximate

Define spectral phase: $\phi(\omega) = \omega n(\omega) L/c$ $\quad$ L is fixed

$\phi(\omega)$ varies slowly away from resonance

Taylor-expand around ω_0

$$\phi(\omega) \approx \phi(\omega_0) + (\omega-\omega_0)\,\phi'(\omega)\Big|_{\omega_0} + \frac{1}{2!}(\omega-\omega_0)^2\,\phi''(\omega)\Big|_{\omega_0} + O(\Delta\omega^3)$$

$\phi(\omega_0) = \omega_0 n(\omega_0) L/c = k_0 n(\omega_0) L$ = constant phase shift. eval @ ω_0

$\phi'(\omega_0)$ = units of time

$\equiv$ "group delay" $= \frac{d}{d\omega}(kL) = \frac{dk}{d\omega} L = L/V_{gr}$

$\phi''(\omega_0)$ = group delay dispersion.

keep 1st order, transform back to time domain

$$\tilde{E}_{out}(\omega) = \tilde{E}_{in}(\omega)\, e^{ik_0 n_0 L}\, e^{i(\omega-\omega_0)\phi'(\omega_0)}$$

$$E_{out}(t) = e^{ik_0 n_0 L}\, \mathcal{F}^{-1}\left\{ \tilde{E}_{in}(\omega)\, e^{i(\omega-\omega_0)\phi'(\omega_0)} \right\}$$

shift theorem:

$$E_{out}(t) = e^{ik_0 n_0 L}\, E_0 \tau\sqrt{\pi}\, \mathcal{F}^{-1}\left\{ e^{-(\omega-\omega_0)^2\tau^2/4}\, e^{i(\omega-\omega_0)\phi'(\omega_0)} \right\}$$

from $\mathcal{F}^{-1}\{F(\omega-\omega_0)\} = f(t)\, e^{-i\omega_0 t}$

let $\omega' = \omega - \omega_0$

$$E_{out}(t) = E_0 \tau\sqrt{\pi}\, e^{i(k_0 n_0 L - \omega_0 t)}\, \mathcal{F}^{-1}\left\{ e^{-\omega'^2\tau^2/4}\, e^{i\omega'\phi'(\omega_0)} \right\}$$

we know that

$$\mathcal{F}\left\{ e^{-t^2/\tau^2} \right\} = \tau\sqrt{\pi}\, e^{-\omega^2\tau^2/4}$$ transform pair for Gaussian

$$\therefore\ \mathcal{F}^{-1}\left\{ e^{-\omega^2\tau^2/4} \right\} = \frac{1}{\tau\sqrt{\pi}}\, e^{-t^2/\tau^2}$$

also, we use shift theorem on inverse transform:

$$\mathcal{F}^{-1}\left\{ F(\omega)\, e^{i\omega t_0} \right\} = f(t-t_0)$$

here $t_0 = \phi'(\omega_0)$ (group delay)

$$E_{out}(t) = E_0\, e^{i(k_0 n_0 L - \omega_0 t)}\, e^{-(t-t_0)^2/\tau^2}$$

$$t_0 = \frac{d}{d\omega}\left(k_0 n(\omega)\right) L = L/v_g$$

w/ $v_g \equiv d\omega/dk$

output pulse is:

$$E_{out}(t) = E_{in}(t - L/v_g)\, e^{i(k_0 n_0 L)}$$

our coordinate system moves at c

so that w/o dispersion, $E_{out}(t) = E_{in}(t)\, e^{i k_0 n_0 L}$

group delay $\tau_g(\omega) = \phi'(\omega)$ arrival time of a group near ω

note that any information must be communicated with some modulation

→ concept that $v_g < c$

even though $v_{ph} = \omega/k$ can be $> c$

Since τ_g varies w/ ω we will see some broadening of the pulse in addition to shift

- must keep next term in expansion of $\phi(\omega)$ to account for this.