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Theorem 1 (First Fundamental Theorem of Calculus). *Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F be the function defined, for all $x \in [a, b]$, by*

$$F(x) = \int_a^x f(t)\,dt.$$

Then, F is differentiable on $[a, b]$, and for every $x \in [a, b]$,

$$F'(x) = f(x).$$

The operation $\int_a^x f(t)\,dt$ is a definite integral with variable upper limit, and its result $F(x)$ is one of the infinitely many antiderivatives of f.

Proof. Let there be two numbers x_1 and $x_1 + \Delta x$ in $[a, b]$. So we have

$$F(x_1) = \int_a^{x_1} f(t)\,dt$$

and

$$F(x_1 + \Delta x) = \int_a^{x_1+\Delta x} f(t)\,dt.$$

Subtracting the two equations gives

$$F(x_1 + \Delta x) - F(x_1) = \int_a^{x_1+\Delta x} f(t)\,dt - \int_a^{x_1} f(t)\,dt. \tag{1}$$

It can be shown that

$$\int_a^{x_1} f(t)\,dt + \int_{x_1}^{x_1+\Delta x} f(t)\,dt = \int_a^{x_1+\Delta x} f(t)\,dt.$$

The sum of the areas of two adjacent regions is equal to the area of both regions combined.

Manipulating this equation gives

$$\int_a^{x_1+\Delta x} f(t)\,dt - \int_a^{x_1} f(t)\,dt = \int_{x_1}^{x_1+\Delta x} f(t)\,dt.$$

Substituting the above into (1) results in

$$F(x_1 + \Delta x) - F(x_1) = \int_{x_1}^{x_1+\Delta x} f(t)\,dt. \tag{2}$$

According to the Intermediate Value Theorem[1] for integration, there exists a $c \in [x_1, x_1 + \Delta x]$ such that

$$\int_{x_1}^{x_1+\Delta x} f(t)\, dt = f(c)\Delta x\,.$$

Substituting the above into (2) we get

$$F(x_1 + \Delta x) - F(x_1) = f(c)\Delta x\,.$$

Dividing both sides by Δx gives

$$\frac{F(x_1 + \Delta x) - F(x_1)}{\Delta x} = f(c).$$

Notice that the expression on the left side of the equation is Newton's difference quotient[2] for F at x_1.

Take the limit as $\Delta x \to 0$ on both sides of the equation.

$$\lim_{\Delta x \to 0} \frac{F(x_1 + \Delta x) - F(x_1)}{\Delta x} = \lim_{\Delta x \to 0} f(c).$$

The expression on the left side of the equation is the definition of the derivative of F at x_1.

$$F'(x_1) = \lim_{\Delta x \to 0} f(c). \tag{3}$$

To find the other limit, we will use the Squeeze Theorem[3]. The number c is in the interval $[x_1, x_1 + \Delta x]$, so $x_1 \leq c \leq x_1 + \Delta x$.
Also, $\lim\limits_{\Delta x \to 0} x_1 = x_1$ and $\lim\limits_{\Delta x \to 0} x_1 + \Delta x = x_1$.
Therefore, according to the squeeze theorem,

$$\lim_{\Delta x \to 0} c = x_1\,.$$

Substituting into (3), we get

$$F'(x_1) = \lim_{c \to x_1} f(c)\,.$$

The function f is continuous at c, so the limit can be taken inside the function. Therefore, we get

$$F'(x_1) = f(x_1)\,.$$

which completes the proof. $\square$

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[1]**Intermediate Value Theorem** states if $y = f(x)$ is continuous on $[a, b]$, and N is a number between $f(a)$ and $f(b)$, then there is a $c \in [a, b]$ such that $f(c) = N$

[2]The function difference divided by the point difference is known as the **difference quotient** and given by $\frac{\Delta F(P)}{\Delta P} = \frac{F(P+\Delta P)-F(P)}{\Delta P}$

[3]The Squeeze Theorem asserts that if two functions approach the same limit at a point, and if a third function is "squeezed" between those functions, then the third function also approaches that limit at that point.