Class 7

- EM wave review
- Calculation of intensity
- Monochromatic Michelson interferometer
- Quasi-monochromatic Michelson
- Autocorrelation theorem
- Fourier Transform interferometer

Solutions of scalar wave equation

- 2nd order PDE: $\frac{\partial^2}{\partial z^2}\psi(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(z,t) = 0$
  - Assume separable solution
  - 2 solutions for f(z), g(t) $\qquad \psi(z,t) = f(z)g(t)$
  - Full solution is a linear combination of both solutions

$$\psi(z,t) = f(z)g(t) = \left(A_1 \cos kz + A_2 \sin kz\right)\left(B_1 \cos \omega t + B_2 \sin \omega t\right)$$

  - Equivalent representation:

$$\psi(z,t) = A_1 \cos\left(kz + \omega t + \phi_1\right) + A_2 \cos\left(kz - \omega t + \phi_2\right)$$

forward propagating + backward propagating waves

- Complex (phasor) representation:

$$\psi(z,t) = \mathrm{Re}\left[a\, e^{i\left(kz - \omega t + \phi\right)}\right] \quad \text{or} \quad \psi(z,t) = \mathrm{Re}\left[A\, e^{i\left(kz - \omega t\right)}\right]$$

Here *A* is complex, includes phase

Maxwell's Equations to wave eqn

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla}\cdot\mathbf{E}=0 \qquad \vec{\nabla}\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\vec{\nabla}\cdot\mathbf{B}=0 \qquad \vec{\nabla}\times\mathbf{B}=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}+\mu_0\frac{\partial\mathbf{P}}{\partial t}$$

Take the curl:

$$\vec{\nabla}\times\left(\vec{\nabla}\times\mathbf{E}\right)=-\frac{\partial}{\partial t}\vec{\nabla}\times\mathbf{B}=-\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}+\mu_0\frac{\partial\mathbf{P}}{\partial t}\right)$$

Use the vector ID:

$$\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)=\mathbf{B}\left(\mathbf{A}\cdot\mathbf{C}\right)-\mathbf{C}\left(\mathbf{A}\cdot\mathbf{B}\right)$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\mathbf{E}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\mathbf{E}\right)-\left(\vec{\nabla}\cdot\vec{\nabla}\right)\mathbf{E}=-\vec{\nabla}^2\mathbf{E}$$

$$\boxed{\vec{\nabla}^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=\mu_0\frac{\partial^2\mathbf{P}}{\partial t^2}}$$

"Inhomogeneous Wave Equation"

Maxwell's Equations in a Medium

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla}^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \mu_0\frac{\partial^2\mathbf{P}}{\partial t^2}$$

- Sinusoidal waves of all frequencies are solutions to the wave equation
- The polarization (**P**) can be thought of as the driving term for the solution to this equation, so the polarization determines which frequencies will occur.
- For linear response, **P** will oscillate at the same frequency as the input.

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0\chi\mathbf{E}$$

- In nonlinear optics, the induced polarization is more complicated:

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0\left(\chi^{(1)}\mathbf{E} + \chi^{(2)}\mathbf{E}^2 + \chi^{(3)}\mathbf{E}^3 + \ldots\right)$$

- The extra nonlinear terms can lead to new frequencies.

Solving the wave equation: linear induced polarization

For low irradiances, the polarization is proportional to the incident field:

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi \mathbf{E}, \quad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1+\chi)\mathbf{E} = \varepsilon \mathbf{E} = n^2 \mathbf{E}$$

In this simple (and most common) case, the wave equation becomes:

$$\vec{\nabla}^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{c^2}\chi \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \rightarrow \vec{\nabla}^2 \mathbf{E} - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Using: $\varepsilon_0 \mu_0 = 1/c^2$ $\qquad \varepsilon_0(1+\chi) = \varepsilon = n^2$

The electric field is a vector function in 3D, so this is actually 3 equations:

$$\vec{\nabla}^2 E_x(\mathbf{r},t) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} E_x(\mathbf{r},t) = 0$$

$$\vec{\nabla}^2 E_y(\mathbf{r},t) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} E_y(\mathbf{r},t) = 0$$

$$\vec{\nabla}^2 E_z(\mathbf{r},t) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} E_z(\mathbf{r},t) = 0$$

Plane wave solutions for the wave equation

If we assume the solution has no dependence on x or y:

$$\vec{\nabla}^2 \mathbf{E}(z,t) = \frac{\partial^2}{\partial x^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial y^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial z^2}\mathbf{E}(z,t) = \frac{\partial^2}{\partial z^2}\mathbf{E}(z,t)$$

$$\rightarrow \frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

The solutions are oscillating functions, for example

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(k_z z - \omega t)$$

Where $\omega = kc, \quad k = 2\pi n/\lambda, \quad v_{ph} = c/n$

This is a *linearly* polarized wave.
For a plane wave **E** is perpendicular to **k**, so **E** can also point in y-direction

Complex notation for EM waves

- Write cosine in terms of exponential

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(kz - \omega t + \phi) = \hat{\mathbf{x}} E_x \frac{1}{2}\left(e^{i(kz-\omega t+\phi)} + e^{-i(kz-\omega t+\phi)}\right)$$

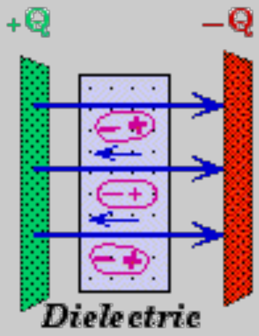
  – Note E-field is a *real* quantity.
    - It is convenient to work with just one component
  – Method 1: $\mathbf{E}(z,t) = \hat{\mathbf{x}}\,\mathrm{Re}\left[A e^{i(kz-\omega t)}\right]$ $\quad A = E_x e^{i\phi}$
  – Method 2: $\mathbf{E}(z,t) = \hat{\mathbf{x}}\left(A e^{i(kz-\omega t)} + c.c.\right)$ $\quad A = \frac{1}{2} E_x e^{i\phi}$
    - In *nonlinear* optics, we have to explicitly include conjugate term. Leads to extra factor of ½.

Wave energy and intensity

- Both E and H fields have a corresponding energy density (J/m^3)
  - For static fields (e.g. in capacitors) the energy density can be calculated through the work done to set up the field

$$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2$$

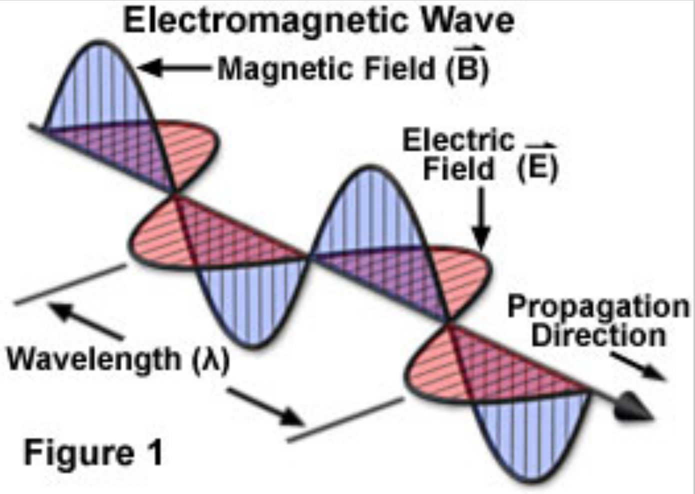
  - Some work is required to polarize the medium
  - Energy is contained in both fields, but H field can be calculated from E field



H field from E field

- H field for a propagating wave is *in phase* with E-field

$$\mathbf{H} = \hat{\mathbf{y}} H_0 \cos\left(k_z z - \omega t\right)$$
$$= \hat{\mathbf{y}} \frac{k_z}{\omega \mu_0} E_0 \cos\left(k_z z - \omega t\right)$$



- Amplitudes are not independent

$$H_0 = \frac{k_z}{\omega \mu_0} E_0 \qquad k_z = n\frac{\omega}{c} \qquad c^2 = \frac{1}{\mu_0 \varepsilon_0} \rightarrow \frac{1}{\mu_0 c} = \varepsilon_0 c$$

$$H_0 = \frac{n}{c \mu_0} E_0 = n \varepsilon_0 c E_0$$

Energy density in an EM wave

- Back to energy density, non-magnetic

$$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu_0 H^2$$

$$H = n\varepsilon_0 cE$$

$$\varepsilon = \varepsilon_0 n^2$$

$$\rho = \tfrac{1}{2}\varepsilon_0 n^2 E^2 + \tfrac{1}{2}\mu_0 n^2 \varepsilon_0{}^2 c^2 E^2$$

$$\mu_0 \varepsilon_0 c^2 = 1$$

$$\rho = \varepsilon_0 n^2 E^2 = \varepsilon_0 n^2 E^2 \cos^2\left(k_z z - \omega t\right)$$

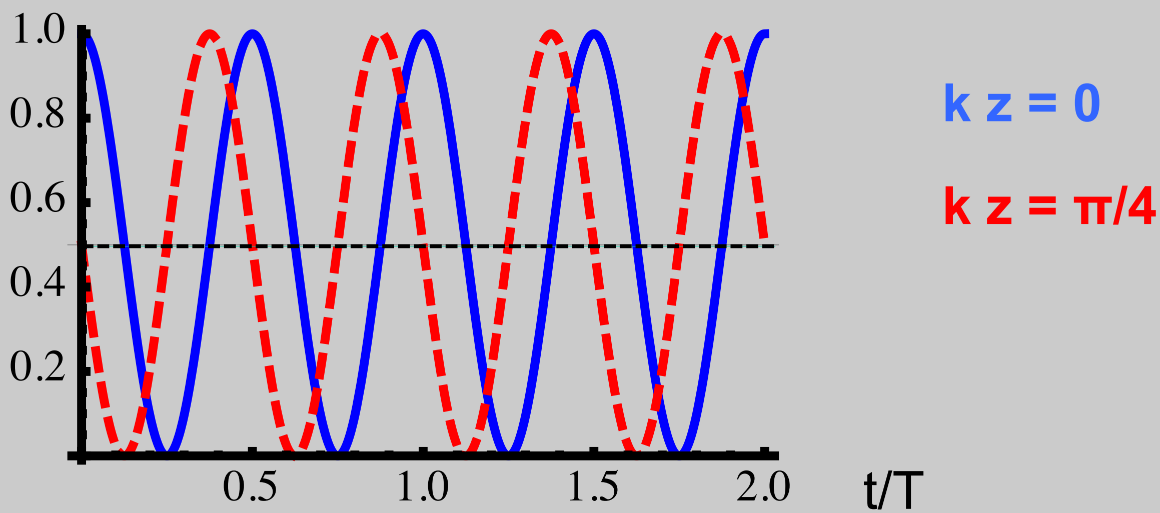
Equal energy in both components of wave

Cycle-averaged energy density

- Optical oscillations are faster than detectors
- Average over one cycle:

$$\langle \rho \rangle = \varepsilon_0 n^2 E_0{}^2 \frac{1}{T}\int_0^T \cos^2\left(k_z z - \omega t\right) dt$$

  – Graphically, we can see this should = ½



  – Regardless of position z

$$\langle \rho \rangle = \frac{1}{2}\varepsilon_0 n^2 E_0{}^2$$

Intensity and the Poynting vector

- Intensity is an energy flux (J/s/cm^2)
- In EM the Poynting vector give energy flux

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

  – For our plane wave,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = E_0 \cos\left(k_z z - \omega t\right) n\varepsilon_0 c E_0 \cos\left(k_z z - \omega t\right) \hat{\mathbf{x}} \times \hat{\mathbf{y}}$$

$$\mathbf{S} = n\varepsilon_0 c E_0^2 \cos^2\left(k_z z - \omega t\right) \hat{\mathbf{z}}$$

  – **S** is along **k**

- Time average: $\mathbf{S} = \frac{1}{2} n \varepsilon_0 c E_0^2 \hat{\mathbf{z}}$
- *Intensity* is the magnitude of **S**

$$I = \frac{1}{2} n\varepsilon_0 c E_0^2 = \frac{c}{n}\rho = V_{phase} \cdot \rho$$

Photon flux: $F = \dfrac{I}{h\nu}$

Calculating intensity with complex wave representation

- Using the convention that we work with the complex form, with the field being the real part

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\,\mathrm{Re}\left[A\,e^{i(kz-\omega t)}\right] \qquad A = E_x e^{i\phi}$$

  – Or write

$$\mathbf{E}(z,t) = \mathbf{E_0} e^{i(kz-\omega t)}$$

$\mathbf{E_0}$ complex, vector

  – take the real part when we want the *field*

- Time-averaged intensity

$$I = \frac{1}{2} n\varepsilon_0 c\,\mathbf{E_0}\cdot\mathbf{E_0}^*$$

  – Notice this is the sum of intensities for the different polarization components

Example: Michelson interferometer

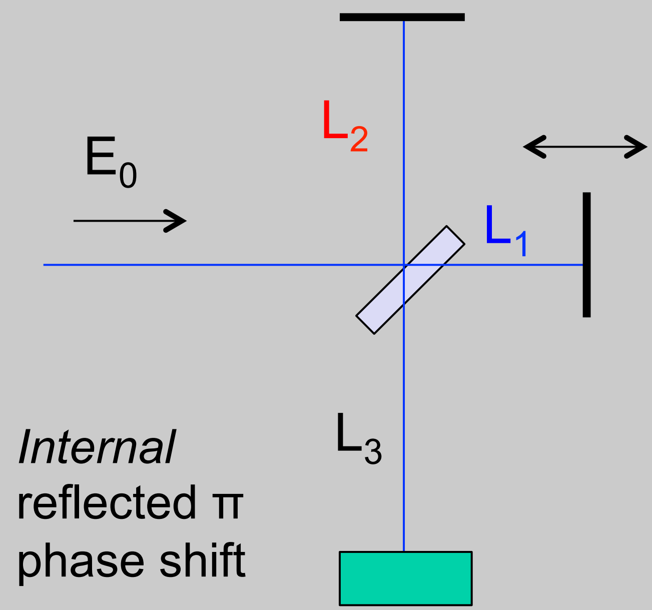
- calculate output intensity
  - 50-50 beamsplitter for *power*
  - Transmitted field:
    - b/s $\frac{1}{\sqrt{2}}\hat{\mathbf{x}}E_0 e^{-i\omega t}$
    - Return $\frac{1}{\sqrt{2}}\hat{\mathbf{x}}E_0 e^{i(2kL_1-\omega t)}$
    - Detector $-\frac{1}{2}\hat{\mathbf{x}}E_0 e^{i\left[k(2L_1+L_3)-\omega t\right]}$
  - Reflected field at detector

$$\frac{1}{2}\hat{\mathbf{x}}E_0 e^{i\left[k(2L_2+L_3)-\omega t\right]}$$

  - Total field at detector

$$\mathbf{E}_{out} = -\frac{1}{2}\hat{\mathbf{x}}E_0 e^{i\left[k(2L_1+L_3)-\omega t\right]} + \frac{1}{2}\hat{\mathbf{x}}E_0 e^{i\left[k(2L_2+L_3)-\omega t\right]}$$

$$= \frac{1}{2}\hat{\mathbf{x}}E_0 e^{i\left[kL_3-\omega t\right]}\left(-e^{ik2L_1} + e^{ik2L_2}\right)$$



Michelson: output intensity

- Calculate intensity of output

$$I = \frac{1}{2} n\varepsilon_0 c\,\mathbf{E}_{out} \cdot \mathbf{E}_{out}^{\ *} = \frac{1}{2} n\varepsilon_0 c\left(\left|\mathbf{E}_1\right|^2 + \left|\mathbf{E}_2\right|^2 + \mathbf{E}_1 \cdot \mathbf{E}_2^{\ *} + \mathbf{E}_2 \cdot \mathbf{E}_1^{\ *}\right)$$

$$\mathbf{E}_{out} = \tfrac{1}{2}\hat{\mathbf{x}} E_0 e^{i\left[k L_3 - \omega t\right]}\left(-e^{ik2L_1} + e^{ik2L_2}\right)$$

$$I = \frac{1}{2} n\varepsilon_0 c\left(\tfrac{1}{2}\hat{\mathbf{x}} E_0 e^{i\left[k L_3 - \omega t\right]}\left(-e^{ik2L_1} + e^{ik2L_2}\right)\right) \cdot \left(\tfrac{1}{2}\hat{\mathbf{x}} E_0 e^{i\left[k L_3 - \omega t\right]}\left(-e^{ik2L_1} + e^{ik2L_2}\right)\right)^*$$

$$I = \frac{1}{8} n\varepsilon_0 c\left|E_0\right|^2 \left(-e^{ik2L_1} + e^{ik2L_2}\right) \cdot \left(-e^{-ik2L_1} + e^{-ik2L_2}\right)$$

In terms of input intensity $I_0 = \frac{1}{2} n\varepsilon_0 c\left|E_0\right|^2$

$$I_{out} = \frac{1}{4} I_0\left(2 - e^{ik2\left(L_1 - L_2\right)} - e^{-ik2\left(L_1 - L_2\right)}\right)$$

$$= \frac{1}{2} I_0\left(1 - \cos\left[k\,2\left(L_1 - L_2\right)\right]\right)$$

In terms of *time delay*

$$2k\left(L_1 - L_2\right) = \omega\,\frac{2\left(L_1 - L_2\right)}{c} = \omega\tau$$

Michelson: time-dependent fields

- Now consider the case where the field has time dependence

$$\mathbf{E}_{in}(t) = \hat{\mathbf{x}} E_0(t) e^{-i\omega_0 t} \qquad \rightarrow \mathbf{E}_{out}(t) = \tfrac{1}{2}\left(\mathbf{E}_{in}(t) - \mathbf{E}_{in}(t-\tau)\right)$$

$$I(t) = \frac{1}{2} n\varepsilon_0 c\left(\left|\mathbf{E}_{in}(t)\right|^2 + \left|\mathbf{E}_{in}(t-\tau)\right|^2 + \mathbf{E}_{in}(t)\cdot\mathbf{E}_{in}(t-\tau)^* + \mathbf{E}_{in}(t-\tau)\cdot\mathbf{E}_{in}(t)^*\right)$$

  – This implicitly is a time average over the fast timescale of the carrier

- Now average over a much longer time

$$\langle I(t)\rangle = \int_{-\infty}^{\infty} I(t)\,dt = 2I_0 + \int_{-\infty}^{\infty} E_0(t) E_0(t-\tau)^*\,dt + c.c.$$

This part is the field autocorrelation

$$E_{AC}(\tau) = \int_{-\infty}^{\infty} E_0(t) E_0^*(t+\tau)\,dt$$

E_{AC} is an even function of τ, so let τ = -τ

Autocorrelation (Wiener-Khinchin) theorem

$$f_{AC}(\tau) = \int f(t) f^*(t+\tau)\, dt$$ autocorrelation

- Connect the autocorrelation to the spectrum

$$FT_\tau\left\{\int f(t) f^*(t+\tau)\,dt\right\} = \int\int f(t) f^*(t+\tau)\,dt\, e^{i\omega\tau}\, d\tau$$

$$= \int f(t)\,dt \int f^*(t+\tau) e^{i\omega\tau}\, d\tau = \int f(t)\,dt \left[\int f(t+\tau) e^{-i\omega\tau}\, d\tau\right]^*$$

Let $t' = t + \tau$ $\quad dt' = d\tau$ $\quad$ But flip limits

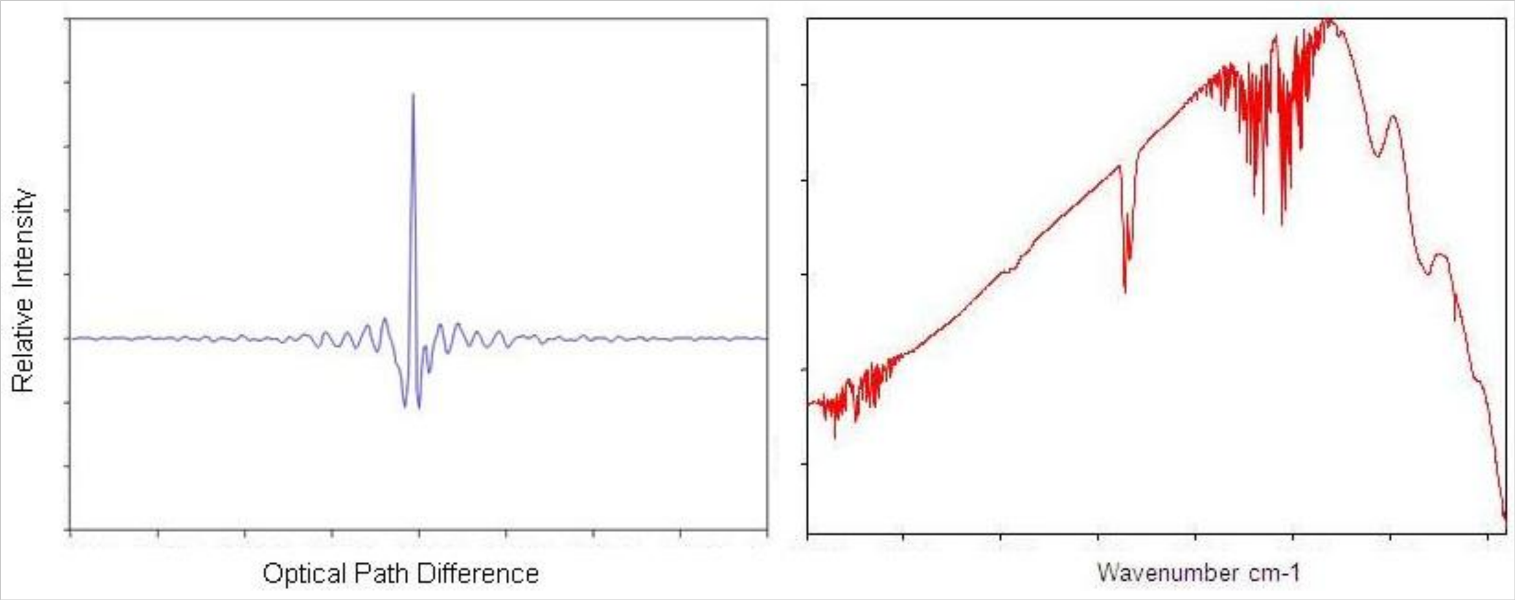
$$FT_\tau\{f_{AC}(t)\} = \int f(t)\,dt\left[\int f(t') e^{-i\omega(t'-t)}\, dt'\right]^* = \int f(t)\,dt\left[F(-\omega)\right]^* e^{-i\omega t}$$

$$= F^*(-\omega)\int f(t) e^{-i\omega t}\, dt = F^*(-\omega) F(-\omega)$$

If f(t) is real, then F(ω) is even, and $$FT_\tau\{f_{AC}(t)\} = \left|F(\omega)\right|^2$$

Fourier transform spectrometer

- Measure interference, subtract DC, FT to get spectrum
  - Single detector, better signal/noise



http://chemwiki.ucdavis.edu/Physical_Chemistry/Spectroscopy/Vibrational_Spectroscopy/Infrared_Spectroscopy/How_an_FTIR_Spectrometer_Operates

Coherence time

- Note that for large time delay, time averaged signal is constant (sum of two intensities)
- Beyond "coherence time" no interference
- Coherence time is inverse of spectral bandwidth

$$T_c \equiv 1/\Delta\nu$$