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Quasi-phase-matching

Gaussian Beam propagation

Quasi-phase matching

- Some materials don't support birefringent phase matching
  - $LiNbO_3$ has a strong NL coefficient but in the same vector direction as the input polarization
  - Isotropic materials, e.g. gas or liquid
- Structuring the medium can allow build-up of NL signal without complete phase matching

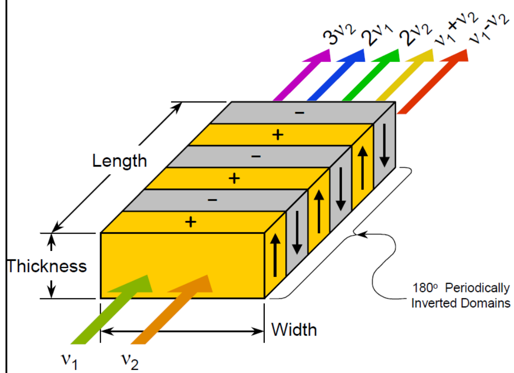
$$\frac{dA_3}{dz} = \frac{2i\,d_{eff}\,\omega_3^2}{k_3 c^2} A_1 A_2 e^{+i\Delta k z} \qquad d_{eff}(z) = d_0 \cos K z$$

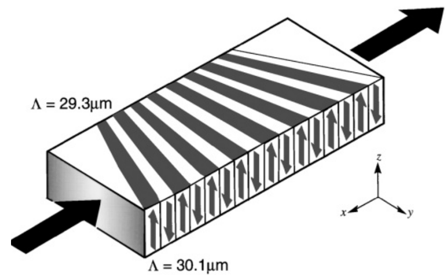
$$\frac{dA_3}{dz} = i\frac{d_0\omega_3^2}{k_3 c^2} A_1 A_2 \left(e^{+iKz} + e^{-iKz}\right) e^{+i\Delta k z}$$

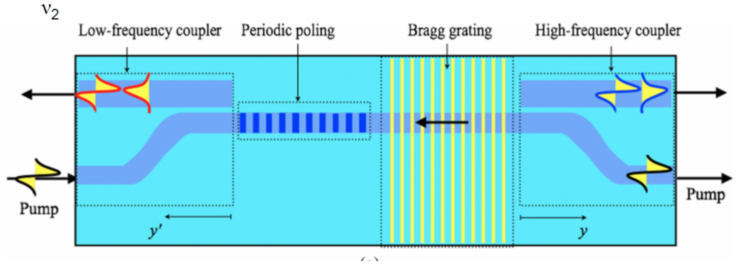
$$= i\frac{d_0\omega_3^2}{k_3 c^2} A_1 A_2 \left(e^{+i(K+\Delta k)z} + e^{-i(K-\Delta k)z}\right)$$

if $K \pm \Delta k = 0$,
signal can build up

PPLN examples

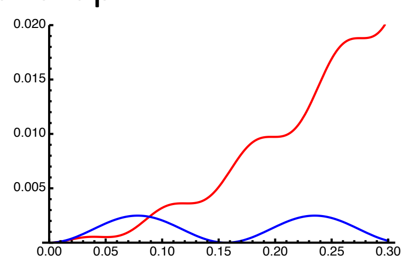




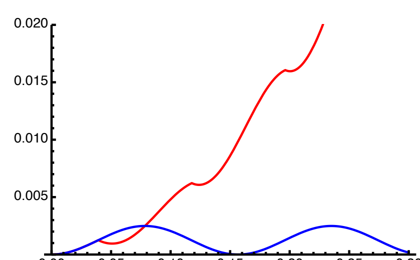


QPM build-up

- QPM cos( ) modulation:



- QPM, Sign(cos()) modulation (layered):
  - Calculate fourier series of modulation function, pick out component that can cancel phase mismatch



3D propagation

$$\nabla^2 \mathbf{E_j} - \frac{n_j^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E_j} = \frac{\partial^2}{\partial z^2}\mathbf{E_j} + \nabla_\perp^2 \mathbf{E_j} - \frac{n_j^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E_j} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\mathbf{P_j}^{NL}$$

- Notes: $\nabla_\perp^2 = \partial_x^2 + \partial_y^2$ $\quad$ $\boxed{\nabla_\perp^2 = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\phi^2}$
  - RHS is source term
  - All linear propagation effects are included in LHS: diffraction, interference, focusing…
- So far, we've assumed plane waves where transverse derivatives are zero.
- Counter examples:
  - Gaussian beams (including high-order)
  - Waveguides
  - Arbitrary propagation
- Often determine solutions to linear equation (e.g. Gaussian beams, waveguide modes), then express fields in terms of those solutions.

Slowly-varying envelope approximation

- Assume waves are forward-propagating:

$$\mathbf{E}_j(\mathbf{r},t) = \mathbf{A}_j(\mathbf{r})e^{i(k_j z - \omega_j t)} + \text{c.c.}$$

$$\mathbf{P}_j(\mathbf{r},t) = \mathbf{p}_j(\mathbf{r})e^{i(k'_j z - \omega_j t)} + \text{c.c.}$$

$$\frac{\partial^2}{\partial z^2}\mathbf{A_j} + 2ik_j\frac{\partial}{\partial z}\mathbf{A_j} - k_j^2\mathbf{A_j} + \nabla_\perp^2\mathbf{A_j} + \frac{n_j^2\omega_j^2}{c^2}\mathbf{A_j} = -\frac{\omega_j^2}{\varepsilon_0 c^2}\mathbf{p_j}e^{i\Delta k z}$$

  - Fast oscillating carrier terms cancel (blue)
- Slowly-varying envelope: compare red terms
  - Drop 2nd order deriv if $\frac{2\pi}{\lambda_j}\frac{1}{L}A_j \gg \frac{1}{L^2}A_j$
  - Ignoring any counterpropagating waves

Diffractive propagation

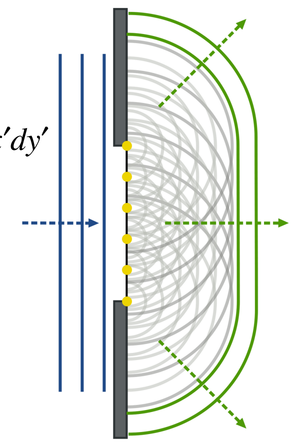
- Huygens' principle:
  - Represent a plane wave as a superposition of source points emitting spherical waves
- Integral representation:

$$E(x,y,z)=\frac{i}{\lambda}\iint E(x',y',z')\frac{\exp\left[-ik\left|\mathbf{r}-\mathbf{r}'\right|\right]}{\left|\mathbf{r}-\mathbf{r}'\right|}\cos\theta\, dx'dy'$$

Field at input plane

Spherical wavelet

Inclination factor



This is essentially a convolution of the complex input field with the spherical wavelets, which are the Green's function for the wave equation

Paraxial approximations

- For **rays**, paraxial = small angle to optical axis
  - Ray slope: $\tan\theta \approx \theta$
- For **spherical waves** where power is directed forward:

$$e^{ikr}=\exp\left[ik\sqrt{x^2+y^2+z^2}\right]$$

$$k\sqrt{x^2+y^2+z^2}=kz\sqrt{1+\frac{x^2+y^2}{z^2}}\approx kz\left(1+\frac{x^2+y^2}{2z^2}\right)$$

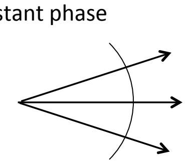
Expanding to 1st order

$$e^{i(kr-\omega t)}\rightarrow e^{ikz}\exp\left[i\left(k\frac{x^2+y^2}{2z}-\omega t\right)\right]$$

z is radius of curvature

Wavefront = surface of constant phase
For x, y >0, t must increase.
Wave is diverging:

$$k\frac{x^2+y^2}{2z}=\omega t$$



Fresnel diffraction integral

- Fresnel approximation (near field)
  - Expand the spherical wave in paraxial approximation (in exponential)
  - Let denominator be $|\mathbf{r} - \mathbf{r}'| \sim z - z' = L \qquad \cos\theta \simeq 1$
  - Input field: $E(x',y',z') = u(x',y',z')e^{-ik(z-z')}$

$$u(x,y,z) = \frac{i}{\lambda L}\iint u(x',y',z')\exp\left[-ik\frac{(x-x')^2+(y-y')^2}{2L}\right]dx'dy'$$

$$u(x,y,z) = \frac{i}{\lambda L}\mathrm{e}^{-ik\frac{x^2+y^2}{2L}}\iint u(x',y',z')\mathrm{e}^{-ik\frac{x'^2+y'^2}{2L}}\mathrm{e}^{-i\frac{k}{L}(xx'+yy')}dx'dy'$$

Gaussian beam solution to wave equation

- Use Fresnel integral to propagate a Gaussian beam

$$U(x,y,z) = \frac{1}{i\lambda z}\mathrm{e}^{ikz}\mathrm{e}^{ik\frac{x^2+y^2}{2z}}\iint\left[e^{-\frac{x'^2+y'^2}{w^2}}\mathrm{e}^{ik\frac{x'^2+y'^2}{2z}}\right]\mathrm{e}^{i2\pi(f_x x'+f_y y')}dx'dy'$$

  - Combine quadratic terms in exponent:

$$\left(-\frac{1}{w^2}+i\frac{k}{2z}\right) = i\frac{k}{2}\left(\frac{1}{z}+i\frac{2}{kw^2}\right) = i\frac{k}{2q}$$

Sign convention for –iωt (opposite Svelto, Seigman)

  - Now integral is a F.T. of a complex Gaussian=Gaussian

$$U(x,y,z) = \frac{i}{i\lambda z}\mathrm{e}^{ik\frac{x^2+y^2}{2z}}\iint\mathrm{e}^{+ik\frac{x'^2+y'^2}{2q}}\mathrm{e}^{i2\pi(f_x x'+f_y y')}dx'\,dy'$$

$$\rightarrow u(r,z) = \frac{1}{q(z)}e^{-ik\frac{r^2}{2q(z)}} \qquad \frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}$$

Complex q form for Gaussian beam

- This combines beam size and radius of curvature into one complex parameter
  - This form is used for for ABCD calculations

$$A(r,z)=A_0\frac{1}{1+i\xi}e^{-\frac{r^2}{w_0^2(1+i\xi)}} \qquad \rightarrow A(r,z)=\frac{1}{q(z)}e^{-ik\frac{r^2}{2q(z)}}$$

$$\frac{1}{q(z)}=\frac{1}{R(z)}-i\frac{\lambda}{\pi w^2(z)}$$

$$\frac{1}{q(z)}=\frac{1}{z+iz_R}=\frac{z}{z^2+z_R^2}-i\frac{z_R}{z^2+z_R^2}$$

$$=\frac{1}{R(z)}-i\frac{w_0^2}{z_R w^2(z)}$$

$$=\frac{1}{R(z)}-i\frac{\lambda}{\pi w^2(z)}=\frac{1}{R(z)}-i\frac{1}{Z(z)}$$

$$\frac{1}{R(z)}=\frac{1}{z\left(1+\frac{z_R^2}{z^2}\right)}=\frac{z}{z^2+z_R^2}$$

$$\frac{1}{w^2(z)}=\frac{1}{w_0^2\left(1+\frac{z^2}{z_R^2}\right)}=\frac{z_R^2}{w_0^2\left(z^2+z_R^2\right)}$$

Complex q into standard form

$$u(r,z)=\frac{1}{q(z)}e^{-ik\frac{r^2}{2q(z)}} \qquad \text{with} \qquad \frac{1}{q(z)}=\frac{1}{R(z)}-i\frac{\lambda}{\pi w^2(z)}$$

Expand exponential:

$$\exp\left[-ik\frac{r^2}{2q(z)}\right]=\exp\left[-ik\frac{r^2}{2}\left(\frac{1}{R(z)}-i\frac{\lambda}{\pi w^2(z)}\right)\right]$$

$$=\exp\left[-ik\frac{r^2}{2}\frac{1}{R(z)}-i\frac{2\pi}{\lambda}\frac{r^2}{2}\left(-i\frac{\lambda}{\pi w^2(z)}\right)\right]=e^{-ik\frac{r^2}{2R(z)}}e^{-\frac{r^2}{w^2(z)}}$$

$$a+ib=\sqrt{a^2+b^2}\,e^{i\arctan(b/a)}$$

Expand leading inverse q:

$$\frac{1}{q(z)}=-i\left(\frac{iz}{z^2+z_R^2}+\frac{z_R}{z^2+z_R^2}\right)=-i\left(\frac{\sqrt{z^2+z_R^2}}{z^2+z_R^2}\right)e^{i\arctan(z/z_R)}$$

$$=-i\left(\frac{1}{z_R\sqrt{1+z^2/z_R^2}}\right)e^{i\arctan(z/z_R)}=\frac{w_0}{iz_R w(z)}e^{i\eta(z)}$$

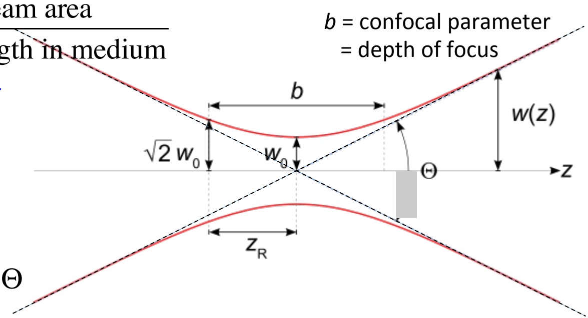
Standard form of Gaussian beam solutions

$$E(r,z,t) = A_0 e^{-i(kz-\omega t)} \frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i\frac{kr^2}{2R(z)}} e^{i\eta(z)}$$

Beam maintains a Gaussian profile as it propagates

- beam radius that varies with z
- Origin of z coordinate is at the beam waist
- Rayleigh length z_R defines collimation distance from focal plane

$$z_R = \frac{\pi w_0^2}{\lambda / n} = \frac{\text{beam area}}{\text{wavelength in medium}}$$



$$w(z) = w_0 \sqrt{1 + \frac{z^2}{z_R^{\,2}}}$$

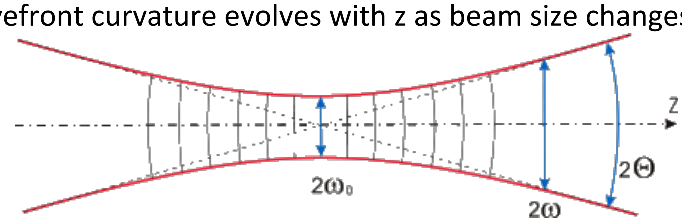
Geometric limit for $z \gg z_R$

$$w(z) = z \frac{w_0}{z_R} = z \tan\Theta$$

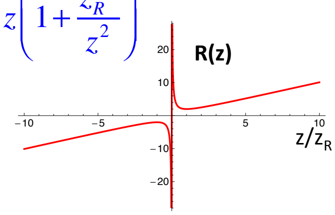
Evolution of wavefronts

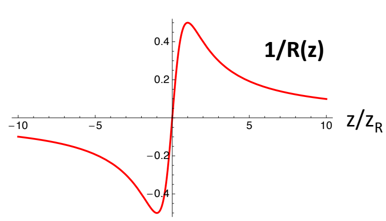
$$E(r,z,t) = A_0 e^{-i(kz-\omega t)} \frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i\frac{kr^2}{2R(z)}} e^{i\eta(z)}$$

- Wavefront curvature evolves with z as beam size changes



$$R(z) = z\left(1 + \frac{z_R^{\,2}}{z^2}\right)$$





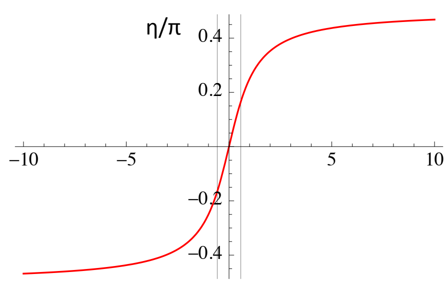
On-axis phase: Gouy phase

$$E(r,z,t) = A_0 e^{-i(kz-\eta(z)-\omega t)} \frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i\frac{kr^2}{2R(z)}}$$

- Because the wavefront changes from focusing to defocusing, on-axis phase advances with z

Gouy phase

$$\eta(z) = \arctan\left(\frac{z}{z_R}\right)$$



Difference between Siegman's complex q and standard form

$$u(r,z) = \frac{1}{q(z)} e^{-ik\frac{r^2}{2q(z)}} = \frac{1}{iz_R} \frac{w_0}{w(z)} e^{i\eta(z)} e^{-ik\frac{r^2}{2R(z)}} e^{-\frac{r^2}{w^2(z)}}$$

$$E(r,z,t) = A_0 \frac{w_0}{w(z)} e^{i(kz-\omega t)} e^{-\frac{r^2}{w^2(z)}} e^{i\frac{kr^2}{2R(z)}} e^{-i\eta(z)}$$

- Siegman's form for the complex q is used almost everywhere for the ABCD calculations.
- He uses the exp[+ iωt] convention, which accounts for the sign difference in the complex exponentials.
- With exp[-iωt] convention, define q as:

$$\frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{\lambda}{\pi w^2(z)} = \frac{1}{z - iz_R}$$

Compare Boyd's form to standard form

- Boyd's complex form is consistent with standard Gaussian beam form

$$A(r,z) = A_0 \frac{1}{1+i\xi} e^{-\frac{r^2}{w_0{}^2(1+i\xi)}} = A_0 \frac{1}{1+iz/z_R} e^{-\frac{r^2}{w_0{}^2(1+iz/z_R)}} \qquad \xi \equiv \frac{z}{z_R}$$

$$\frac{1}{1+i\xi} = \frac{1}{1+iz/z_R} = \frac{z_R}{z_R+iz} = \frac{-iz_R}{z-iz_R} = \frac{-iz_R}{q(z)}$$

$$A(r,z) = A_0(-iz_R)\frac{1}{q(z)} e^{+\frac{iz_R r^2}{w_0{}^2 q(z)}} = -iz_R A_0 \frac{1}{q(z)} e^{+\frac{ikr^2}{2q(z)}}$$