MATH348-Advanced Engineering Mathematics Homework: Fourier Series - Part II : Solutions

Complex Representation, Resonant Forcing, Bessel's Inequality, Parseval's Identity

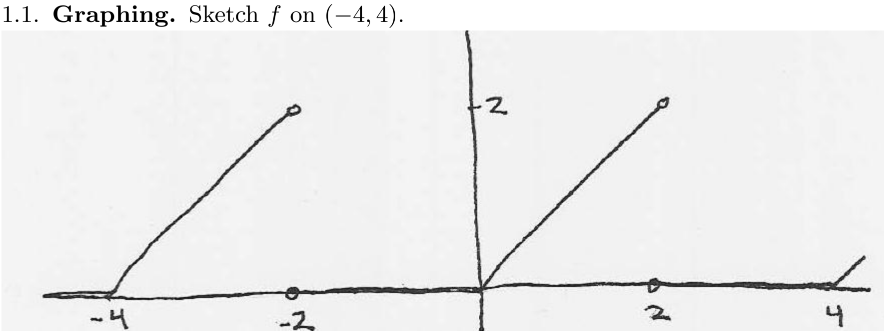
Text: 11.3-11.4 | Lecture Notes : 9-10 | Lecture Slides: N/A

Quote of Fourier Series Homework - Part II : Solutions
Whoever can see through all fear will always be safe.
Tao Te Ching : Laozi (late 4^{th} or early 3^{rd} centuries BC)

1. Fourier Series : Nonstandard Period

Let $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$ be such that $f(x+4) = f(x)$.

1.1. **Graphing.** Sketch f on $(-4, 4)$.



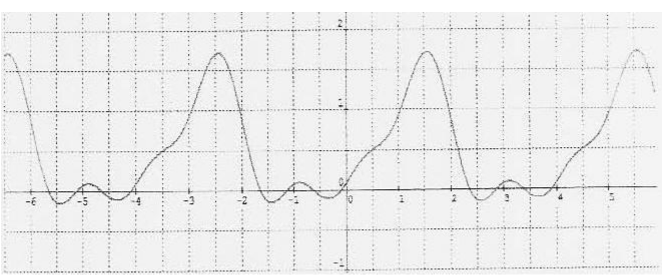
1.2. **Symmetry.** Is the function even, odd or neither?

This function is neither even nor odd.

1.3. **Integrations.** Determine the Fourier coefficients a_0, a_n, b_n of f.

$$
\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-1}^{1} f(x)\,dx = \frac{1}{4}\left[\int_{-2}^{0} 0dx + \int_{0}^{2} xdx\right] = \\
&= \frac{1}{8}x^2\Big|_0^2 = \frac{1}{2} \\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \\
&= \frac{1}{2}\left[\int_{-2}^{0} 0\cdot\cos\left(\frac{n\pi}{2}x\right)dx + \int_{0}^{2} x\cdot\cos\left(\frac{n\pi}{2}x\right)dx\right] = \\
&= \frac{1}{2}\left[\frac{2x}{n\pi}\sin(n\pi 2x) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi}{2}x\right)\right]_0^2 = \\
&= \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos(n\pi) - \frac{4}{n^2\pi^2}\right] = \frac{2(-1)^n - 2}{n^2\pi^2} \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx = \\
&= \frac{1}{2}\left[\int_{-2}^{0} 0\cdot\sin\left(\frac{n\pi}{2}x\right) + \int_{0}^{2} x\cdot\sin\left(\frac{n\pi}{2}x\right)dx\right] = \\
&= \frac{1}{2}\left[\frac{-2x}{n\pi}\cos\left(\frac{n\pi}{2}x\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}x\right)\right]_0^2 = \\
&= \frac{1}{2}\left[\frac{-4}{n\pi}(-1)^n\right] = \frac{2(-1)^{n+1}}{n\pi} \\
f(x) &= \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n - 2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}x\right) + \frac{2(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi}{2}x\right)\right] \\
& a_0 = \frac{1}{2} \quad a_n = \frac{2(-1)^n - 2}{n^2\pi^2} \quad b_n = \frac{2(-1)^{n+1}}{n\pi}
\end{aligned}
$$

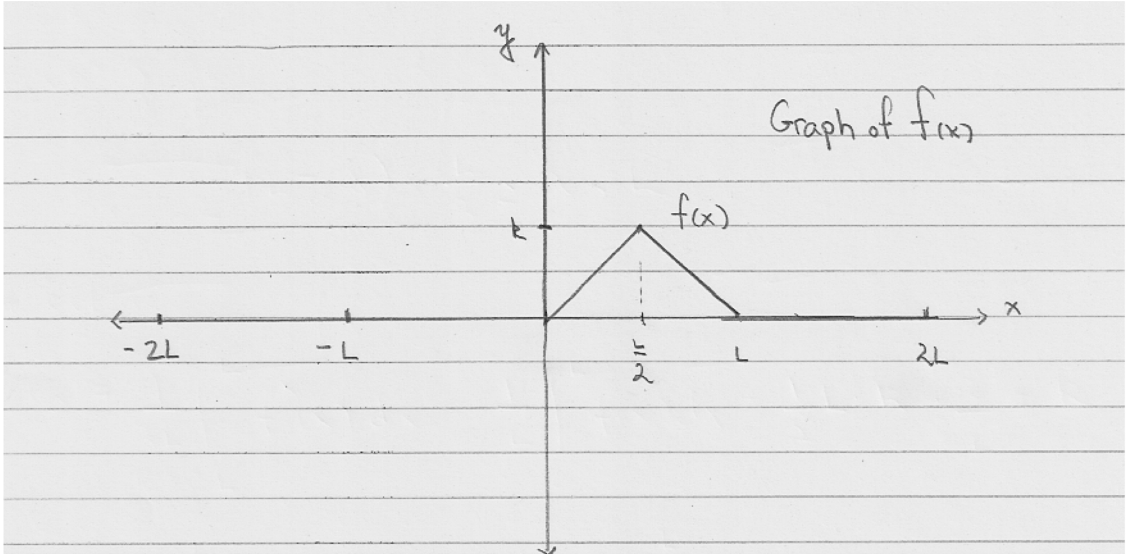
1.4. **Truncation.** Using `http://www.tutor-homework.com/grapher.html`, or any other graphing tool, graph the first five terms of your Fourier Series Representation of f.



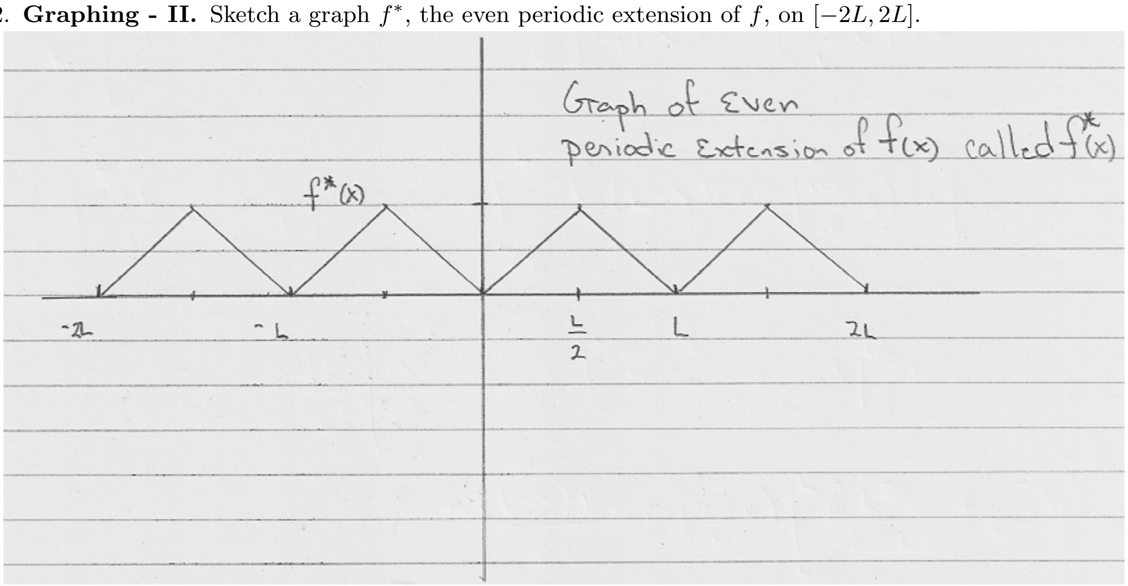
2. Fourier Series : Periodic Extension

Let $f(x) = \begin{cases} \frac{2k}{L}x, & 0 < x \leq \frac{L}{2} \\ \frac{2k}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$.

2.1. **Graphing - I.** Sketch a graph f on $[-2L, 2L]$.



2.2. **Graphing - II.** Sketch a graph f^*, the even periodic extension of f, on $[-2L, 2L]$.



2.3. **Fourier Series.** Calculate the Fourier cosine series for the half-range expansion of f.

We begin with the coefficient a_0 and noting that this is nothing more than the area under the curve $f(x)$ we find,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f^*(x)dx \tag{1}$$

$$= \frac{1}{L}\int_{0}^{L} f(x)dx \tag{2}$$

$$= \frac{1}{L}\cdot\frac{1}{2}Lk \tag{3}$$

$$= \frac{k}{2} \tag{4}$$

Next we have a_n. Some symmetry, integration by parts and algebra gives,

$$a_n = \frac{1}{L}\int_{-L}^{L} f^*(x)\cos\left(\frac{n\pi}{L}x\right)dx \tag{5}$$

$$= \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \tag{6}$$

$$= \frac{2}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\cos\left(\frac{n\pi}{L}x\right)dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\cos\left(\frac{n\pi}{L}x\right)dx\right] \tag{7}$$

$$= \frac{4k}{L^2}\left[\frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\Big|_0^{\frac{L}{2}} + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\Big|_0^{\frac{L}{2}} + \frac{L}{n\pi}(L-x)\sin\left(\frac{n\pi}{L}x\right)\Big|_{\frac{L}{2}}^{L} - \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\Big|_{\frac{L}{2}}^{L}\right] \tag{8}$$

$$= \frac{4k}{L^2}\left[\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}x\right) + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}x\right) - \frac{L^2}{n^2\pi^2} - \frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}x\right) - \frac{L^2}{n^2\pi^2}(-1)^n + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}x\right)\right] \tag{9}$$

$$= \frac{4k}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}x\right) - (-1)^n - 1\right] \tag{10}$$

Further simplifications can be made. If we note the following pattern,

$$n=1 \implies a_1 = 0, \tag{11}$$

$$n=2 \implies a_2 = -\frac{16k}{2^2n^2}, \tag{12}$$

$$n=3 \implies a_3 = 0, \tag{13}$$

$$n=4 \implies a_4 = 0, \tag{14}$$

$$n=5 \implies a_5 = 0, \tag{15}$$

$$n=6 \implies a_6 = -\frac{-16k}{6^2n^2}, \tag{16}$$

we can write the Fourier cosine series as,

$$f(x) = \frac{k}{2} + \sum_{n=1}^{\infty}\frac{4k}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}x\right) - (-1)^n - 1\right]\cos\left(\frac{n\pi}{L}x\right) \tag{17}$$

$$= \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\left(\frac{2\pi}{L}x\right) + \frac{1}{6^2}\cos\left(\frac{6\pi}{L}x\right) + \cdots\right) \tag{18}$$

3. Complex Fourier Series

3.1. **Orthogonality Results.** Show that $\left\langle e^{inx}, e^{-imx}\right\rangle = 2\pi\delta_{nm}$ where $n, m \in \mathbb{Z}$, where $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$.

$$\text{For } n \neq m \quad \int_{-\pi}^{\pi} e^{inx}e^{-imx}dx = \int_{-\pi}^{\pi} e^{(n-m)ix}dx = \frac{e^{(n-m)ix}}{i(n-m)}\Big|_{-\pi}^{\pi} = \tag{19}$$

$$= \frac{(-1)^{(n-m)}}{i(n-m)} - \frac{(-1)^{(n-m)}}{i(n-m)} = 0 \tag{20}$$

$$\text{For } n = m \quad \int_{-\pi}^{\pi} e^{inx}e^{-imx}dx = \int_{-\pi}^{\pi} e^{(n-m)ix}dx = \int_{-\pi}^{\pi} 1dx = x|_{-\pi}^{\pi} = 2\pi \tag{21}$$

$$= \int_{-\pi}^{\pi} e^{inx}e^{-imx}dx = \begin{cases} 0 & n \neq m \\ 2\pi & n = m \end{cases} = 2\pi\delta_{nm} \tag{22}$$

3.2. **Fourier Coefficients.** Using the previous orthogonality relation find the Fourier coefficients, c_n, for the complex Fourier series, $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$.

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \\
&\Rightarrow f(x)e^{-imx} = \sum_{n=-\infty}^{\infty} c_n e^{inx} e^{-imx} \\
&\Rightarrow \int_{-\infty}^{\infty} f(x)e^{-imx} dx = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} c_n e^{(n-m)x} dx \\
&\quad \text{As we found in (a), the integral on the right is 0 for all values of n except n=m} \\
&\Rightarrow \int_{-\pi}^{\pi} f(x)e^{-imx} dx = c_m 2\pi \\
&\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx} dx = c_m \\
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx} dx
\end{aligned}
$$

Because m=n we can replace our m's with n's to get the formula for c_n

3.3. **Complex Fourier Series Representation.** Find the complex Fourier coefficients for $f(x) = x^2,\ -\pi < x < \pi,\ f(x+2\pi) = f(x)$.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{23}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx} dx \tag{24}$$

$$= \frac{1}{2\pi}\left[\frac{-x^2}{in}e^{-inx} + \frac{2x}{n^2}e^{-inx} + \frac{2}{in^3}e^{-inx}\right]_{-\pi}^{\pi} \tag{25}$$

$$= \frac{1}{2\pi}\left[\left(\frac{-x^2}{in} + \frac{2x}{n^2} + \frac{2}{in^3}\right)e^{-inx}\right]_{-\pi}^{\pi} \tag{26}$$

$$= \frac{1}{2\pi}\left[\left(\frac{-\pi^2}{in} + \frac{2\pi}{n^2} + \frac{2}{in^3} + \frac{\pi^2}{in} + \frac{2\pi}{n^2} - \frac{2}{in^3}\right)(-1)^n\right] \tag{27}$$

$$= \frac{1}{2\pi}\left[\frac{4\pi}{n^2}(-1)^n\right] = \frac{2}{n^2}(-1)^n \qquad n \neq 0 \tag{28}$$

$$\text{For } n = 0 \tag{29}$$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{i(0)x} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi}\left[\frac{1}{3}x^2\right]_{-\pi}^{\pi} = \tag{30}$$

$$= \frac{\pi^2}{3} \tag{31}$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{2}{n^2}(-1)^n e^{inx} \tag{32}$$

3.4. **Conversion to Real Fourier Series.** Using the complex Fourier series representation of f recover its associated real Fourier series.

$$f(x) = \frac{\pi^2}{3} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{2}{n^2}(-1)^n e^{inx} \tag{33}$$

$$= \frac{\pi^2}{3} + \sum_{n=-\infty}^{-1} \frac{2}{n^2}(-1)^n e^{inx} + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n e^{inx} \tag{34}$$

(35) Substituting n=-n into the first series we get:

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n e^{-inx} + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n e^{inx} \tag{36}$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n \left(e^{-inx} + e^{inx}\right) \tag{37}$$

(38) Using Euler's Formula:

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n \left(\cos(nx) - i\sin(nx) + \cos(nx) + \sin(nx)\right) \tag{39}$$

(40) The Real Fourier Series Representation:

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(nx) \tag{41}$$

4. Periodic Forcing of Simple Harmonic Oscillators

Consider the ODE, which is commonly used to model forced simple harmonic oscillation,

$$y'' + 9y = f(t), \tag{42}$$

$$f(t) = |t|, \quad -\pi \le t < \pi, \quad f(t+2\pi) = f(t). \tag{43}$$

Since the forcing function (43) is a periodic function we can study (42) by expressing $f(t)$ as a Fourier series. [1] [2]

4.1. **Fourier Series Representation.** Express $f(t)$ as a real Fourier series.

4.2. **Method of Undetermined Coefficients.** The solution to the homogeneous problem associated with (42) is $y_h(t) = c_1\cos(3t) + c_2\sin(3t)$, $c_1, c_2 \in \mathbb{R}$. Knowing this, if you were to use the method of undetermined coefficients[3] then what would your choice for the particular solution, $y_p(t)$? DO NOT SOLVE FOR THE UNKNOWN CONSTANTS

4.3. **Resonant Modes.** What is the particular solution associated with the third Fourier mode of the forcing function?[4]

4.4. **Structural Changes.** What is the long term behavior of the solution to (42) subject to (43)? What if the ODE had the form $y'' + 4y = f(t)$?

5. Error Analysis and Applications

We have that for a *reasonable* $2\pi-$periodic function there exist coefficients a_0, a_n, b_n such that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx). \tag{44}$$

This is, of course, the Fourier series representation of the function f but, as we know, computational devices are not well-suited to infinite sums. Thus, we would like to know how f is approximated by

$$f(x) \approx f_N(x) = a_0 + \sum_{n=1}^{N} a_n\cos(nx) + b_n\sin(nx), \tag{45}$$

were this $N^{th}-$partial sum is called a trigonometric polynomial. Since f_N approximates f on an interval, we define our error as

$$E = \int_{-\pi}^{\pi} (f - f_N)^2 dx, \tag{46}$$

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[1]The advantage of expressing $f(t)$ as a Fourier series is its validity for any time t. An alternative approach have been to construct a solution over each interval $n\pi < t < (n+1)\pi$ and then piece together the final solution assuming that the solution and its first derivative is continuous at each $t = n\pi$.

[2]It is worth noting that this concepts are used by structural engineers, a sub-disciple of civil engineering, to study the effects of periodic forcing on buildings and bridges. In fact, this problem originate from a textbook on structural engineering.

[3]This is also known as the method of the 'lucky guess' in your differential equations text.

[4]Each term in a Fourier series is called a mode. The first mode is sometimes called the fundamental mode. The rest of the modes, called *harmonics* in acoustics, are just referenced by number. The third Fourier mode would be the third term of Fourier summation

which is called *the squared error* of f_N.[5] It can be shown that this squared error can be written as

$$E = \int_{-\pi}^{\pi} f^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right]. \tag{47}$$

It is plausible that as $\lim_{N\to\infty} f_N = f$ and $E \to 0$. Thus, from (47) we have

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2 dx, \tag{48}$$

which is called Parseval's identity.[6]

5.1. **Application of Mean Square Error.** Let $f(x) = x^2$ for $x \in (-\pi, \pi)$ such that $f(x + 2\pi) = f(x)$. Determine the value of N so that $E < 0.001$.

Application of the previous formulae gives that,

$$E = \int_{\pi}^{\pi} x^4 dx - \pi \left[2\frac{\pi^4}{9} + \sum_{n=1}^{N} \frac{16}{n^4} \right] \tag{49}$$

$$= \frac{8\pi^5}{45} - 16\pi \sum_{n=1}^{N} \frac{1}{n^4} \tag{50}$$

Now we must figure out what N value will make $E < 0.001$. Excel can do this but I have created a mathematica notebook, that is linked to the blog that can do it as well. Here is a screenshot of the results.

```
For N=1 the error is 4.13802
For N=2 the error is 0.996424
For N=3 the error is 0.375863
For N=4 the error is 0.179513
For N=5 the error is 0.0990886
For N=6 the error is 0.0603035
For N=7 the error is 0.0393683
For N=8 the error is 0.0270964
For N=9 the error is 0.0194352
For N=10 the error is 0.0144086
For N=11 the error is 0.0109754
For N=12 the error is 0.00855134
For N=13 the error is 0.00679141
For N=14 the error is 0.00548296
For N=15 the error is 0.00449006
For N=16 the error is 0.00372307
For N=17 the error is 0.00312124
For N=18 the error is 0.00264241
For N=19 the error is 0.0022567
For N=20 the error is 0.00194254
For N=21 the error is 0.00168409
For N=22 the error is 0.00146951
For N=23 the error is 0.00128989
For N=24 the error is 0.00113838
For N=25 the error is 0.0010097
For N=26 the error is 0.000899709
For N=27 the error is 0.000805126
For N=28 the error is 0.000723347
For N=29 the error is 0.000652279
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So, it seems like when $N = 26$ the error drops below 0.001.

5.2. **Application of Parseval's identity.** Using the previous function and Parseval's identity show that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

[5] We choose to square the integrand so that there can be no possible cancellation of positive errors/areas with negative errors/areas.

[6] These are the main equations associated with the error analysis of Fourier series. A student interested in the derivations should consult Kreyszig's section 11.4, 9th edition.

We know that as $N \to \infty$ the error goes to zero. Thus from equation (49) we have that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{8\pi^5}{45} \frac{1}{16\pi} = \frac{\pi^4}{90}, \tag{51}$$

which is the desired result.