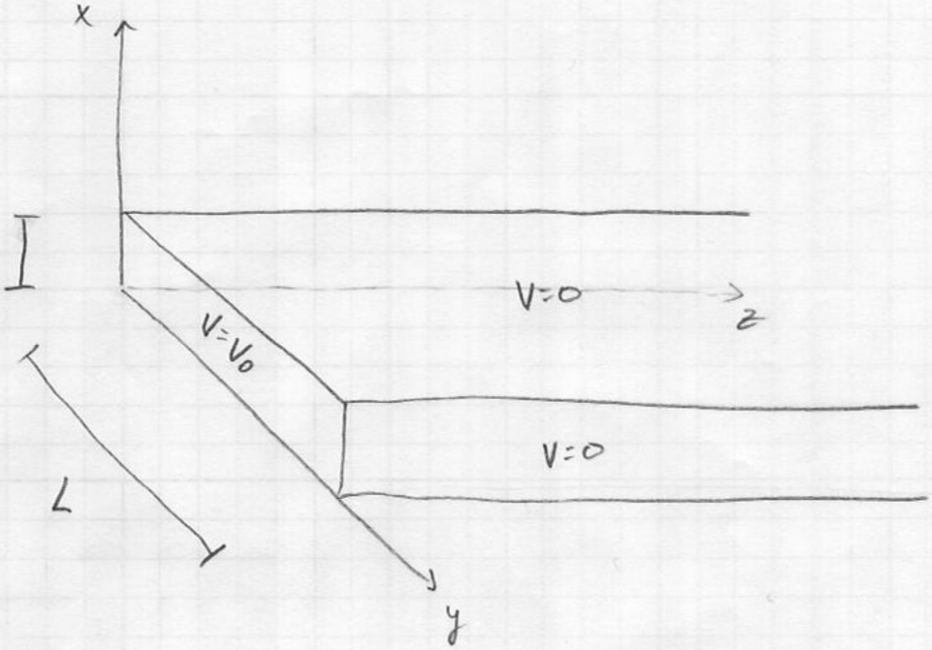
Phys 361 Recitation - Separation of variables

1) a)



The equation that fully describes electrostatics is Poisson's equation:

$$-\nabla^2 V = \rho/\varepsilon_0$$

But if we're searching for the potential inside the hollow pipe and nowhere else, $\rho=0$ everywhere we're looking and Poisson's equation reduces to Laplace's equation:

$$\nabla^2 V = 0 \qquad \text{or, in Cartesian,}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad (1)$$

b) The cap at $z=0$ breaks translational symmetry in that direction, and x and y don't behave any better. We're solving our equations in all three variables.

c) I'll guess that $V(x,y,z) = X(x)\,Y(y)\,Z(z)$

and I'll substitute that into (1).

$$\frac{d^2X}{dx^2}YZ + X\frac{d^2Y}{dy^2}Z + XY\frac{dZ}{dz} = 0$$

Now I'll divide by XYZ:

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\mathrm{I}} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\mathrm{II}} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\mathrm{III}} = 0$$

d) Note that I, II, and III are each functions of single variables. That requires that each be equal to some constant.

How can I be so sure? Well, suppose $\frac{1}{X}\frac{d^2X}{dx^2}$ did vary with x. II and III definitely don't vary with x, so if I fiddle with x, I'll change I without changing II and III, and I'll change their sum. But

I + II + III has to equal zero for all x, y, and z for

us to satisfy Laplace's equation. So we'd have a contradiction.

The only way to obtain a solution everywhere is to have

$$\frac{1}{X}\frac{d^2X}{dx^2} = \kappa_1^2 \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = \kappa_2^2 \qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = \kappa_3^2$$

for some kappas such that $\boxed{\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 0}$

2a) Differential equations of the form $X'' = K_1^2 X$

have solutions of the form $e^{\pm K_1 x}$, where K_1 might be real or imaginary depending on the sign of K_1^2. So solutions might include things like

$$\boxed{e^{kx}, \quad e^{-kx}, \quad e^{ikx}, \quad e^{-ikx}}$$

note $\Longrightarrow$ where $k = |K_1|$

We also know that linear combinations of these are solutions, and appropriate combos of real exponentials form hyperbolic trig functions, while combos of complex exponentials yield regular trig functions, so we might also use:

$$\boxed{\cosh(kx), \quad \sinh(kx), \quad \cos(kx), \quad \sin(kx)}$$

and their counterparts in y and z.

b) Let's think about z first. At $z=0$, $V(x,y,0) = V_0$.

Then, V should drop monotonically to zero as $z \to \infty$. After all, very far away, there are grounded plates all around us, and no sources to be seen.

sin & cos are periodic, so they won't be much help

$\sinh(z)$ & $\cosh(z)$ + e^z diverge as z increases, so they're even less help. Only functions of the form $\boxed{e^{-k_3 z}}$

can possibly yield a legitimate solution in z.

What about in x? We need $V=0$ at $x=0$ and $x=W$.

Getting zeros in two places with one of those places being the origin is something that only sine functions can deliver.

Same argument holds in y.

Thus X must go like $\sin(k_1 x)$ and Y like $\sin(k_2 y)$

A solution of the form $e^{-k_3 z}$ comes from a positive K_3^2,

and sines come from negative K_1^2 and K_2^2, so our choices are consistent with the requirement that

$K_1^2 + K_2^2 + K_3^2$ needs to be able to be zero.

And it <u>will</u> be zero as long as $k_1^2 + k_2^2 = k_3^2$, so we only have to find two of those and we'll automatically know the third.

c) Our full series solution should therefore look like:

$$V(x,y,z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{mn} e^{-k_{3mn} z} \sin(k_{1n} x) \sin(k_{2m} y)$$

Since only two of k_1, k_2, and k_3 are actually independent, you only need a double sum. Each value of n yields a new k_1, and likewise for k_2 and m, but once you pick those, k_3 is set in stone. No third degree of freedom.

3a) Ok, so $V(x,y,z)$ must be zero for all y and z when $x=0$ or $x=W$. If that has to be true for all y and z, it must be something we accomplish purely with the x term, so:

$$\sin(K_{1n}\cdot 0) = \sin(K_{1n}W) = 0$$

The first of those isn't helpful - $\sin(0)=0$ no matter what.

But the second is helpful. it tells us $K_{1n}W$ must equal some integer multiple of π:

$$K_{1n}W = n\pi \Rightarrow \boxed{K_{1n} = \frac{n\pi}{W}}$$

Identical arguments in the y direction yield $\boxed{K_{2m} = \frac{m\pi}{L}}$

Which means $\boxed{K_{3mn} = \sqrt{\left(\frac{n\pi}{W}\right)^2 + \left(\frac{m\pi}{L}\right)^2}}$

b) Our last condition is that $V=V_0$ at $z=0$ for all x and y. At $z=0$, $e^{-K_{3mn}z}=1$, so we get:

$$V(x,y,0) = V_0 = \sum_{n=1}\sum_{m=1} C_{mn}\sin\left(\frac{n\pi x}{W}\right)\sin\left(\frac{m\pi y}{L}\right)$$

We can extract the C_{mn} using Fourier's trick in 2D. Multiply both sides by $\sin\left(\frac{l\pi x}{W}\right)\sin\left(\frac{j\pi y}{L}\right)$ and integrate to exploit orthogonality:

$$\int_0^L\int_0^W V_0 \sin\left(\frac{l\pi x}{W}\right)\sin\left(\frac{j\pi y}{L}\right)dx\,dy =$$

$$\sum_n\sum_m \int_0^L\int_0^W C_{mn}\sin\left(\frac{l\pi x}{W}\right)\sin\left(\frac{n\pi x}{W}\right)\sin\left(\frac{j\pi y}{L}\right)\sin\left(\frac{m\pi y}{L}\right)dx\,dy$$

And we know from math phys (take a drink):

$$\int_0^W \sin\left(\frac{\ell\pi x}{W}\right)\sin\left(\frac{n\pi x}{W}\right)dx = \delta_{\ell n}\frac{W}{2}$$

$$\int_0^L \sin\left(\frac{j\pi y}{L}\right)\sin\left(\frac{m\pi y}{L}\right)dy = \delta_{jm}\frac{L}{2}$$

The Kronecker deltas collapse the sums, giving us

$$V_0\int_0^L \sin\left(\frac{j\pi y}{L}\right)dy\int_0^W \sin\left(\frac{\ell\pi x}{W}\right)dx = C_{\ell j}\cdot\frac{W}{2}\frac{L}{2}$$

And $\displaystyle\int_0^L \sin\left(\frac{j\pi y}{L}\right)dy = -\frac{L}{\pi j}\cos\left(\frac{j\pi y}{L}\right)\Big|_0^L$

$$= \frac{-L}{\pi j}\Big[\cos(j\pi) - \cos(0)\Big]$$

Here's a tricky part: When $j=0$, $\cos(j\pi)=1$ and the integral yields 0. But when $j=1$, $\cos(j\pi)=-1$ and the integral gives not-zero. This pattern continues. For even j we get zero, for odd we get

$$= -\frac{L}{\pi j}\big[-1-1\big] = \frac{2L}{\pi j}$$

Therefore $\displaystyle\int_0^W \sin\left(\frac{\ell\pi x}{W}\right)dx = \frac{2W}{\pi\ell}$ if ℓ is odd, 0 if even

And so

$$V_0\cdot\frac{2W}{\pi\ell}\cdot\frac{2L}{\pi j} = \frac{W}{2}\frac{L}{2}C_{\ell j} \Rightarrow C_{\ell j} = \frac{16V_0}{\pi^2\ell j}$$ if ℓ and j are both odd, zero otherwise

c) Putting it all together and relabeling indices:

$$\boxed{V(x,y,z) = \sum_{n=1,3,5\ldots}^{\infty}\sum_{m=1,3,5\ldots}^{\infty}\frac{16V_0}{\pi^2 nm}e^{-\sqrt{\left(\frac{n\pi}{W}\right)^2+\left(\frac{m\pi}{L}\right)^2}\,z}\sin\left(\frac{n\pi x}{W}\right)\sin\left(\frac{m\pi y}{L}\right)}$$

That is beefy.

4a) This part is rather subjective and answers may vary. I look at it and see an exponential decay in z (a good thing) and some sine functions that'll guarantee that the solutions will "fit" in the pipe. It's strongly reminiscent of the particle in a box from quantum mechanics, except that the complete summed solution will be fairly smooth instead of varying like a single sine function.

b) We have a bunch of terms in the series, all of the form

$$e^{-k_3 z} \sin(k_1 x) \sin(k_2 y)$$

And since $k_3 = \sqrt{\left(\frac{n\pi}{w}\right)^2 + \left(\frac{m\pi}{L}\right)^2}$, the exponent in that $e^{-k_3 z}$ term is increasing. Something that looks like (for example) e^{-2z} drops to zero significantly faster than e^{-z}, so unless we're at fairly small z, the lowest-order term in the series should be dominant. Somewhat, anyway.

c) Answers may vary. See attached.

It's tough to graph a function of three variables directly and get something that's easy to look at. In this problem, though, x and y are on more-or-less equal status. The x and y portions of the solution have the same form, with the only difference being the L or W. We could probably get a decent look at the essentials of the solution by suppressing either x or y entirely. Another option would be to plot with respect to two of the variables (x and z, for example) several times at different y values, to get cross sections. Let's do the latter and see what happens. I'll get everything fully defined and set L and W to one (they're just arbitrary scale factors):

In[31]:=
```
L := 1;  W := 1
```

In[32]:=
```
k3[n_, m_] := Sqrt[(n * π / W) ^2 + (m * π / L) ^2]
```

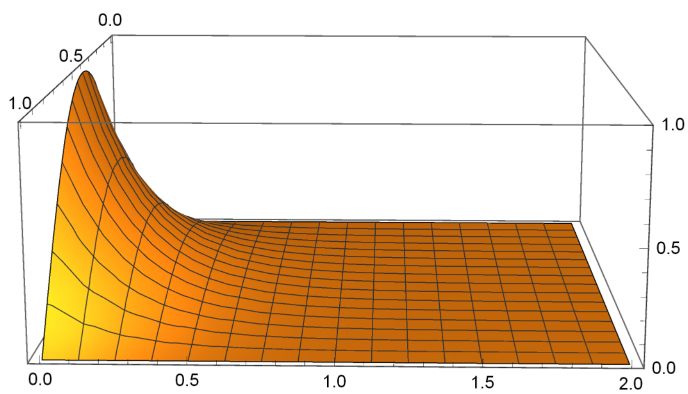
In[33]:=
```
Vsingle[x_, y_, z_, n_, m_] := Exp[-k3[n, m] * z] * Sin[n * π * x / W] * Sin[m * π * y / L]
```

In[34]:=
```
Vtotal[x_, y_, z_, p_] := Sum[Vsingle[x, y, z, n, m], {n, 1, p, 2}, {m, 1, p, 2}]
```

And now I'll graph a single term at some particular y, but not y = 0.

In[35]:=
```
Plot3D[Vtotal[x, 0.5, z, 1], {x, 0, W}, {z, 0, 2}, PlotRange → {0, 1}]
```

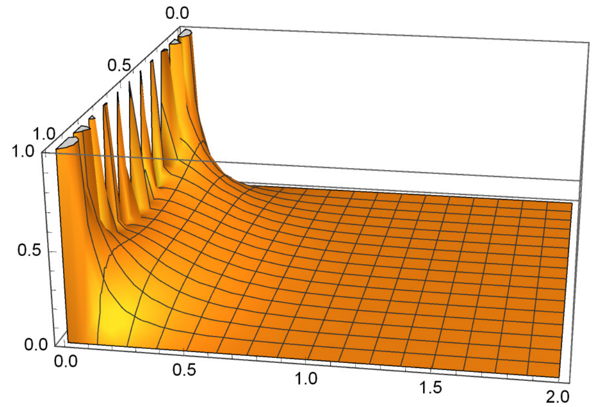
Out[35]=



Perhaps not that surprisingly, that drops to zero in a wicked hurry. But it doesn't seem to fit the endcap boundary condition very well (that the voltage at z = 0 should be constant). That's certainly what we'd expect to break the worst if we only kept a single term of the series. Let's see what it looks like if we keep a bunch more terms:

In[36]:= `Plot3D[Vtotal[x, 0.5, z, 21], {x, 0, W}, {z, 0, 2}, PlotRange → {0, 1}]`

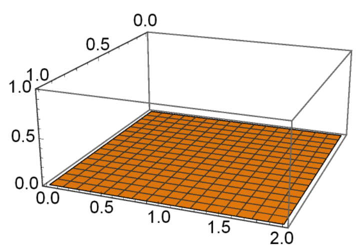
Out[36]=



Visibly different, especially at very small z, but not that different for mid-sized z. It looks like the sum is doing a better job of matching that endcap condition, too.
Now let’s take a few slices at different y:

In[37]:= `Plot3D[Vtotal[x, 0.0, z, 21], {x, 0, W}, {z, 0, 2}, PlotRange → {0, 1}]`

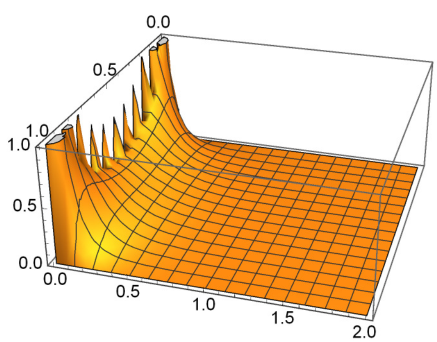
Out[37]=



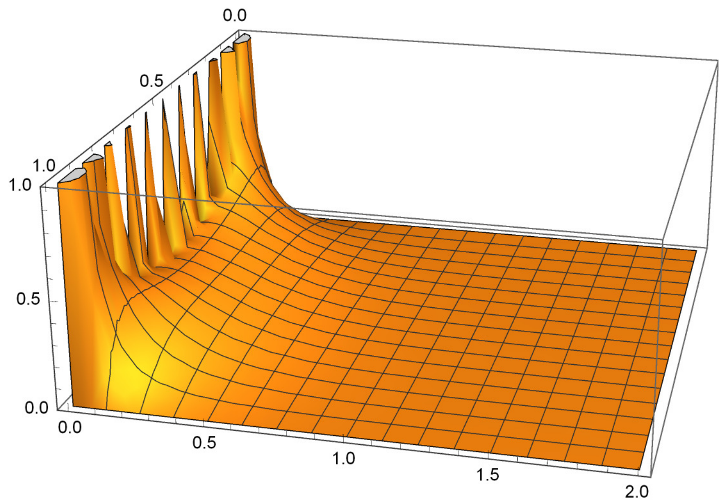
At y = 0 we get zero voltage everywhere, as we should.

In[38]:= `Plot3D[Vtotal[x, 0.25, z, 21], {x, 0, W}, {z, 0, 2}, PlotRange → {0, 1}]`

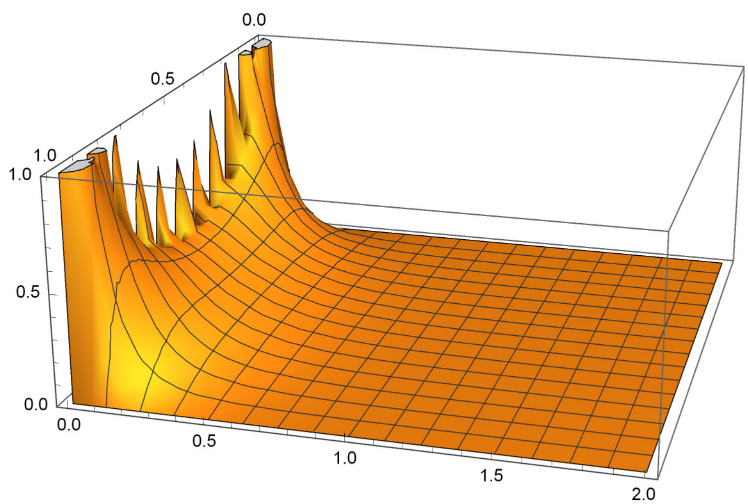
Out[38]=



In[39]:=
```
Plot3D[Vtotal[x, 0.5, z, 21], {x, 0, W}, {z, 0, 2}, PlotRange → {0, 1}]
```

Out[39]=


In[40]:=
```
Plot3D[Vtotal[x, 0.75, z, 21], {x, 0, W}, {z, 0, 2}, PlotRange → {0, 1}]
```

Out[40]=


Followed by a not-terribly-exciting progression.