MATH 348 - Advanced Engineering Mathematics April 17, 2009

Homework 10, Spring 2009 **Due**:April 29th, 2009

PARTIAL DIFFERENTIAL EQUATIONS - HEAT AND WAVE EQUATIONS

Consider the one-dimensional heat equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \tag{1}$$

$$x \in (0, L), \quad t \in (0, \infty), \quad c^2 = \frac{K}{\sigma \rho}. \tag{2}$$

Equations (1)-(2) model the time-evolution of the temperature, $u = u(x,t)$, of a heat conducting medium in one-dimension. The object, of length L, is assumed to have a homogenous thermal conductivity K, specific heat σ, and linear density ρ. That is, $K, \sigma, \rho \in \mathbb{R}^+$.

1. Consider the one-dimensional heat equation, (1)-(2), with the boundary conditions[1],

$$u_x(0,t) = 0, u_x(L,t) = 0, \tag{3}$$

$$u(x,0) = f(x). \tag{4}$$

   (a) Assume that the solution to (1)-(2) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (1)-(2), which satisfies (3)-(4). [2]

   (b) Describe how the long term behavior of the general solution to (1)-(4) changes as the thermal conductivity, K, is increased while all other parameters are held constant. Also, describe how the solution changes when the linear density, ρ, is increased while all other parameters are held constant.

   Define,

$$f(x) = \begin{cases} \dfrac{2k}{L} x, & 0 < x \leq \frac{L}{2} \\ \frac{2k}{L}(L - x), & \frac{L}{2} < x < L \end{cases} \tag{5}$$

   and for the following questions we consider the solution, u, to the heat equation given by, (1)-(2), which satisfies the initial condition given by (5). [3]

   (c) For $L = 1$ and $k = 1$, find the particular solution to (1)-(2) with boundary conditions (3)-(4) for when the initial temperature profile of the medium is given by (5). Show that $\lim_{t \to \infty} u(x,t) = f_{avg} = 0.5$.[4]

2. Recall the 1-D conservation law encountered during the derivation of the heat equation.[5]

$$\frac{\partial u}{\partial t} = -\kappa \frac{\partial \phi}{\partial x}, \quad \kappa \in \mathbb{R} \tag{6}$$

   In general, if the function $u = u(x,t)$ represents the density of a physical quantity then the function $\phi = \phi(x,t)$ represents its flux. If we assume the ϕ is proportional to the negative gradient of u then, from (6), we get the one-dimensional heat/diffusion equation (1).[6]

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[1] Here the boundary conditions correspond to perfect insulation of both endpoints

[2] An insulated bar is discussed in examples 4 and 5 on page 557.

[3] When solving the following problems it would be a good idea to go back through your notes and the homework looking for similar calculations.

[4] It is interesting here to note that though the initial condition f doesn't appear to satisfy the boundary conditions its periodic Fourier extension does. That is, if you draw the even periodic extension of the initial condition then you would see that the slope is not well defined at the end points. Remembering that the Fourier series averages the right and left hand limits of the periodic extension of the function f at the endpoints shows that the boundary conditions are, in fact, satisfied, since the derivative of an average is the average of derivatives.

[5] When discussing heat transfer this is known as Fourier's Law of Cooling. In problems of steady-state linear diffusion this would be called Fick's First Law. In discussing electricity u could be charge density and q would be its flux.

[6] AKA Fick's Second Law associated with linear non-steady-state diffusion.

(a) Assume that ϕ is proportional to u to derive, from (6), the convection/transport equation, $u_t + cu_x = 0$, where c is some proportionality constant.

(b) Given the initial condition $u(x,0) = u_0(x)$ for the convection equation, show that $u(x,t) = u_0(x-ct)$ is a solution to this PDE.

(c) If both diffusion and convection are present in the physical system then the flux is given by, $\phi(x,t) = cu - du_x$, where $c, d \in \mathbb{R}^+$. Derive from, (6), the convection-diffusion equation $u_t + cu_x - du_{xx} = 0$.

(d) If there is also energy/particle loss proportional to the amount present then we introduce to the convection-diffusion equation the term λu to get the convection-diffusion-decay equation,[7]

$$u_t = Du_{xx} - cu_x - \lambda u. \tag{7}$$

Show that by assuming, $u(x,t) = w(x,t)e^{\alpha x - \beta t}$, (7) can be transformed into a heat equation on the new variable w where $\alpha = c/(2D)$ and $\beta = \lambda + c^2/(4D)$.[8]

Consider the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \tag{8}$$

$$x \in (0, L)\,, \qquad t \in (0, \infty)\,, \qquad c^2 = \frac{T}{\rho}. \tag{9}$$

Equations (1)-(2) model the time-evolution of the displacement, $u = u(x,t)$, of an elastic medium in one-dimension. The object, of length L, is assumed to have a homogenous lateral tension T, and linear density ρ. That is, $T, \rho \in \mathbb{R}^+$.

3. Consider the one-dimensional wave equation, (8)-(9), with the boundary conditions[9],

$$u_x(0,t) = 0, u_x(L,t) = 0, \tag{10}$$

and initial conditions,

$$u(x,0) = f(x), \tag{11}$$

$$u_t(x,0) = g(x). \tag{12}$$

(a) Assume that the solution to (8)-(9) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (8)-(9), which satisfies (10)-(12). [10] [11]

(b) Let $L = 1$ and $k = 1$ and find the particular solution, which satisfies the initial displacement, $f(x)$, given by (5) and has zero initial velocity for all points on the object.

4. Show that by direct substitution that the function $u(x,t)$ given by,

$$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(y)dy, \tag{13}$$

is a solution to the one-dimensional wave equation where u_0 and v_0 are the initial displacement and velocity of the elastic string, respectively. [12]

[7] The u_{xx} term models diffusion of energy/particles while u_x models convection, u models energy/particle loss/decay.

[8] This shows that the general PDE (7) can be solved using heat equation techniques.

[9] These boundary conditions imply that the object must have zero curvature at each endpoint.

[10] It is important to notice that the solution to the spatial portion of the problem is the same as the heat problem of hw5 prob2.

[11] Remember that in this case we have nontrivial solutions for $k_0 = 0$. You should find that $G_0(t) = C_1 + C_2 t$.

[12] This is called the D'Alembert solution to the wave equation. To do this you may want to recall the fundamental theorem of calculus, $\frac{d}{dx}\int_0^x f(t)dt = f(x)$ and properties of integrals, $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$.

5. Consider the non-homogenous one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x,t) \quad , \tag{14}$$

$$x \in (0, L)\,, \qquad t \in (0, \infty)\,, \qquad c^2 = \frac{T}{\rho}. \tag{15}$$

with boundary conditions and initial conditions,

$$u(0,t) = u(L,t) = 0, \tag{16}$$

$$u(x,0) = u_t(x,0) = 0. \tag{17}$$

Letting $F(x,t) = A\sin(\omega t)$ gives the following Fourier Series Representation of the forcing function F ,

$$F(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{L}\right), \tag{18}$$

where

$$f_n(t) = \frac{2A}{n\pi}(1 - (-1)^n)\sin(\omega t). \tag{19}$$

(a) Show that substitution of (18)-(19) into (14) yields the ODE,

$$G_n'' + \left(\frac{cn\pi}{L}\right)^2 G_n = \frac{2A}{n\pi}(1 - (-1)^n)\sin(\omega t). \tag{20}$$

(b) The solution to (20) is given by,

$$G_n(t) = G_n^h(t) + G_n^p(t), \tag{21}$$

where $G_n^h(t) = B_n \cos\left(\frac{cn\pi}{L}x\right) + B_n^* \sin\left(\frac{cn\pi}{L}x\right)$ is the homogenous solution and $G_n^p(t)$ is the particular solution to (20).

i. If $\omega \neq cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS

ii. If $\omega = cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS

iii. For the latter case what is $\lim_{t\to\infty} u(x,t)$ and what does this limit imply physically?