MATH 332 - Linear Algebra June 23, 2009
Homework 3, Summer 2009 **Due**: June 29, 2009

Linear Independence - Matrix Transformation - Matrix Operations

1. Given that,

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix},$$

and observe that the first column plus twice the second column equals the third column. Find a nontrivial solution of $\mathbf{Ax} = \mathbf{0}$.
Hint: Row reduction will find nontrivial solutions to the system. However, it is unnecessary to use row reduction.

2. Determine the values of h for which the vectors are linearly dependent.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$$

3. Suppose the vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose, as well, that $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, \ldots, p$. Show that T is the zero-transformation. That is, show that if $\mathbf{x}$ is any vector in $\mathbb{R}^n$, then $T(\mathbf{x}) = \mathbf{0}$.
Hint: If $\mathbf{v_1}, \ldots, \mathbf{v_p}$ generates $\mathbb{R}^n$ then how can **any** vector in $\mathbb{R}^n$ be written?

4. In quantum mechanics spin one-half particles, typically an electron[1] , can be characterized by the following vectors:

$$\mathbf{e}_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $\mathbf{e}_u$ represents spin-up and $\mathbf{e}_d$ represents spin-down.[2] The following matrices:

$$\mathbf{S}_+ = \hbar \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{S}_- = \hbar \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

where $\hbar$ is Planck's constant, are linear transformations, which act on $\mathbf{e}_u$ and $\mathbf{e}_d$.

(a) Compute and describe the effect of the transformations, $\mathbf{S}_+(\mathbf{e}_u + \mathbf{e}_d)$, and $\mathbf{S}_-(\mathbf{e}_u + \mathbf{e}_d)$.

(b) $\mathbf{S}_+$ and $\mathbf{S}_-$ are projection transformations. Projection transformations are known to destroy information. Justify this in the case of $\mathbf{S}_+$ and $\mathbf{S}_-$ by showing that for any vector $\mathbf{b} \in \mathbb{R}^2$ there does NOT exist a **unique** $\mathbf{x} \in \mathbb{R}^2$, which satisfies $\mathbf{S}_+(\mathbf{x}) = \mathbf{b}$ or $\mathbf{S}_-(\mathbf{x}) = \mathbf{b}$.

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[1] In general these particles are called fermions. `http://en.wikipedia.org/wiki/Spin-1/2`, `http://en.wikipedia.org/wiki/Fermions`

[2] In quantum mechanics, the concept of spin was originally considered to be the rotation of an elementary particle about its own axis and thus was considered analogous to classical angular momentum subject to quantum quantization. However, this analogue is only correct in the sense that spin obeys the same rules as quantized angular momentum. In 'reality' spin is an intrinsic property of elementary particles and it is the roll of quantum mechanics to understand how to associate quantized particles with spin to their associated background field in such a way that certain field properties/symmetries are preserved. This is studied in so-called quantum field theory. `http://www.physics.thetangentbundle.net/wiki/Quantum_mechanics/spin`, `http://en.wikipedia.org/wiki/Spin_(physics)`, `http://en.wikipedia.org/wiki/Quantum_field_theory`

5. We define the commutator and anti-commutation functions on matrices as[3],

$$[A, B] = AB - BA, \quad \{A, B\} = AB + BA. \tag{1}$$

The following matrices are the so-called Pauli spin matrices and have interesting commutation and anti-commutation relations and gives us fine setting to practice our matrix algebra. [4]

$$\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2}$$

Using the previous definitions show the following:

(a) $\sigma_i^2 = \mathbf{I}$ for $i = 1, 2, 3$. [5]

(b) $[\sigma_i, \sigma_j] = 2i \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. [6]

(c) $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{I}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. [7]

[3] A commutator is a function, which takes in two matrices and returns one and is, in some sense, a measure of the binary operations lack of commutativity.

[4] The Pauli spin matrices are a set of Hermitian matrices, which are *unitary.* They have found several uses including describing strong interaction symmetries in particle physics, logic gates in quantum information theory and representation of finite groups in abstract algebra.

[5] This statement encapsulates both the symmetric unitary properties of the matrices.

[6] Here we are using the so-called Levi-Civita symbol. This symbol is used to encode the following commonly encountered information,

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1) \text{ or } (3,1,2), \\ -1, & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2) \text{ or } (2,1,3), \\ 0, & \text{if } i = j \text{ or } j = k \text{ or } k = i \end{cases} \tag{3}$$

[7] Here we use the so-called Kronecker delta function, which encodes the, also common, information,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases} \tag{4}$$