Pulse characterization

- How do we know when the pulse is short?
- Can we measure the phase of a pulse?

- Field autocorrelation
- Intensity autocorrelation
  - Collinear
  - Non-collinear
- Frequency-resolved optical gating
- Dispersion scan

Example: Michelson interferometer

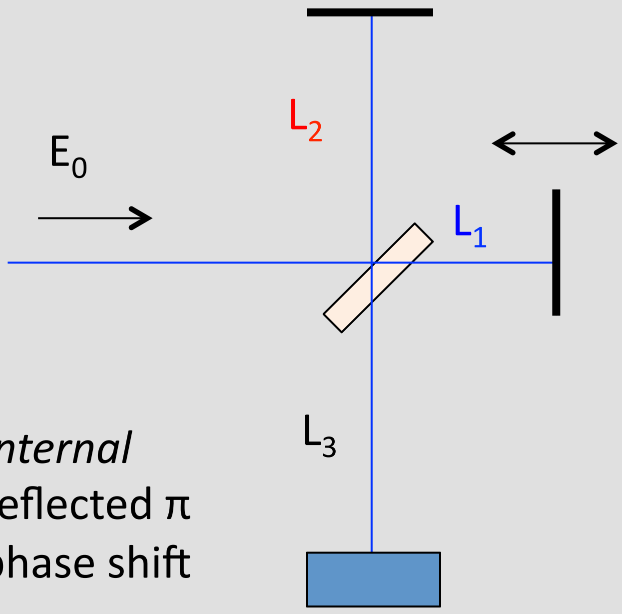
- calculate output intensity
  - 50-50 beamsplitter for *power*
  - Transmitted field:
    - b/s $\frac{1}{\sqrt{2}}\hat{\mathbf{x}}E_0 e^{-i\omega t}$
    - Return $\frac{1}{\sqrt{2}}\hat{\mathbf{x}}E_0 e^{i(2kL_1-\omega t)}$
    - Detector $-\frac{1}{2}\hat{\mathbf{x}}E_0 e^{i[k(2L_1+L_3)-\omega t]}$
  - Reflected field at detector

$$\frac{1}{2}\hat{\mathbf{x}}E_0 e^{i[k(2L_2+L_3)-\omega t]}$$

  - Total field at detector

$$\mathbf{E}_{out} = -\frac{1}{2}\hat{\mathbf{x}}E_0 e^{i[k(2L_1+L_3)-\omega t]} + \frac{1}{2}\hat{\mathbf{x}}E_0 e^{i[k(2L_2+L_3)-\omega t]}$$
$$= \frac{1}{2}\hat{\mathbf{x}}E_0 e^{i[kL_3-\omega t]}\left(-e^{ik2L_1} + e^{ik2L_2}\right)$$

Internal reflected π phase shift



Michelson: output intensity

- Calculate intensity of output

$$I = \frac{1}{2} n\varepsilon_0 c\,\mathbf{E}_{out} \cdot \mathbf{E}_{out}{}^* = \frac{1}{2} n\varepsilon_0 c\left(\left|\mathbf{E}_1\right|^2 + \left|\mathbf{E}_2\right|^2 + \mathbf{E}_1 \cdot \mathbf{E}_2{}^* + \mathbf{E}_2 \cdot \mathbf{E}_1{}^*\right)$$

$$\mathbf{E}_{out} = \tfrac{1}{2}\hat{\mathbf{x}} E_0 e^{i[kL_3 - \omega t]}\left(-e^{ik2L_1} + e^{ik2L_2}\right)$$

$$I = \frac{1}{2} n\varepsilon_0 c\left(\tfrac{1}{2}\hat{\mathbf{x}} E_0 e^{i[kL_3 - \omega t]}\left(-e^{ik2L_1} + e^{ik2L_2}\right)\right) \cdot \left(\tfrac{1}{2}\hat{\mathbf{x}} E_0 e^{i[kL_3 - \omega t]}\left(-e^{ik2L_1} + e^{ik2L_2}\right)\right)^*$$

$$I = \frac{1}{8} n\varepsilon_0 c\left|E_0\right|^2 \left(-e^{ik2L_1} + e^{ik2L_2}\right) \cdot \left(-e^{-ik2L_1} + e^{-ik2L_2}\right)$$

In terms of input intensity $I_0 = \frac{1}{2} n\varepsilon_0 c\left|E_0\right|^2$

$$I_{out} = \frac{1}{4} I_0\left(2 - e^{ik2(L_1 - L_2)} - e^{-ik2(L_1 - L_2)}\right)$$

$$= \frac{1}{2} I_0\left(1 - \cos\left[k\,2\left(L_1 - L_2\right)\right]\right)$$

In terms of *time delay*

$$2k\left(L_1 - L_2\right) = \omega \frac{2\left(L_1 - L_2\right)}{c} = \omega\tau$$

Michelson: time-dependent fields

- Now consider the case where the field has time dependence

$$\mathbf{E}_{in}(t) = \hat{\mathbf{x}} E_0(t) e^{-i\omega_0 t} \qquad \rightarrow \mathbf{E}_{out}(t) = \tfrac{1}{2}\left(\mathbf{E}_{in}(t) - \mathbf{E}_{in}(t-\tau)\right)$$

$$I(t) = \frac{1}{2} n\varepsilon_0 c \left(\left|\mathbf{E}_{in}(t)\right|^2 + \left|\mathbf{E}_{in}(t-\tau)\right|^2 + \mathbf{E}_{in}(t)\cdot\mathbf{E}_{in}(t-\tau)^* + \mathbf{E}_{in}(t-\tau)\cdot\mathbf{E}_{in}(t)^* \right)$$

  – This implicitly is a time average over the fast timescale of the carrier

- Now average over a much longer time

$$\left\langle I(t) \right\rangle = \int_{-\infty}^{\infty} I(t)\,dt = 2I_0 + \int_{-\infty}^{\infty} E_0(t) E_0(t-\tau)^* \,dt + c.c.$$

This part is the field autocorrelation

$$E_{AC}(\tau) = \int_{-\infty}^{\infty} E_0(t) E_0^{\,*}(t+\tau)\,dt$$

E_{AC} is an even function of τ, so let τ = -τ

Autocorrelation (Wiener-Khinchin) theorem

$$f_{AC}(\tau)=\int f(t)f^*(t+\tau)dt$$ autocorrelation

- Connect the autocorrelation to the spectrum

$$FT_\tau\left\{\int f(t)f^*(t+\tau)dt\right\}=\int\int f(t)f^*(t+\tau)dt\,e^{i\omega\tau}\,d\tau$$

$$=\int f(t)dt\int f^*(t+\tau)e^{i\omega\tau}\,d\tau=\int f(t)dt\left[\int f(t+\tau)e^{-i\omega\tau}\,d\tau\right]^*$$

Let $t'=t+\tau$ $dt'=d\tau$ But flip limits

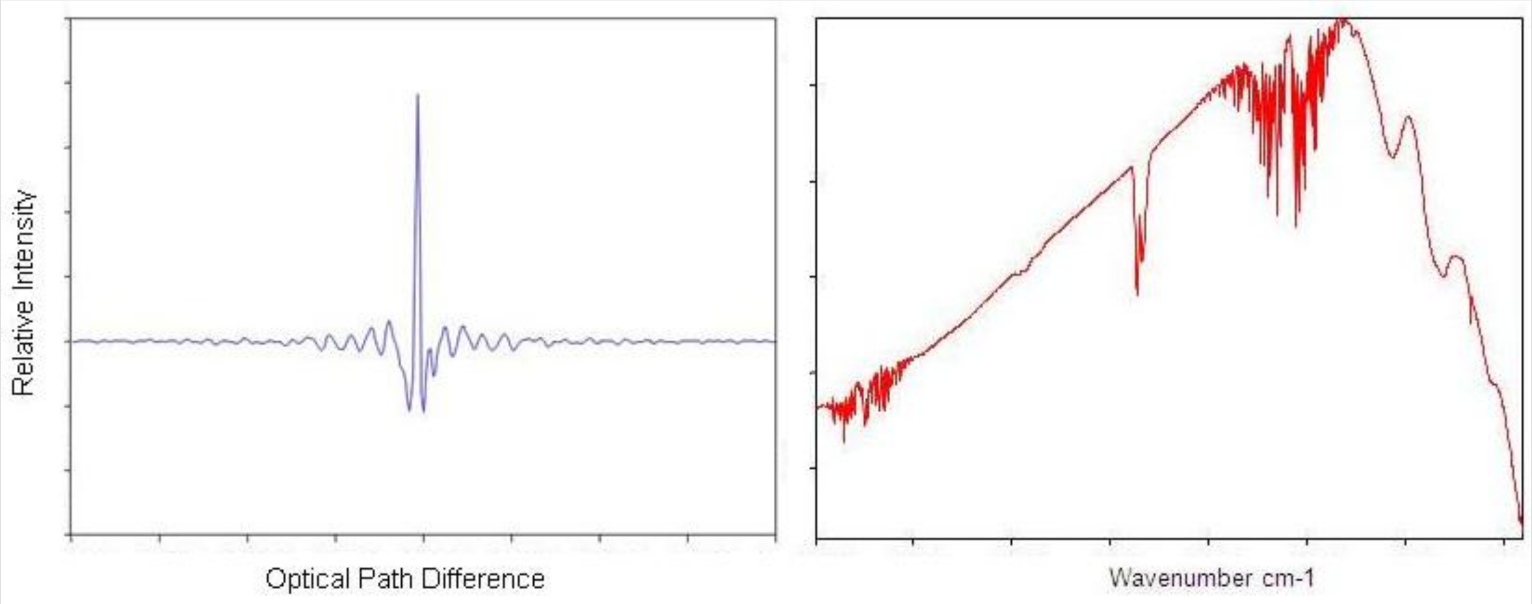
$$FT_\tau\left\{f_{AC}(t)\right\}=\int f(t)dt\left[\int f(t')e^{-i\omega(t'-t)}\,dt'\right]^*=\int f(t)dt\left[F(-\omega)\right]^*e^{-i\omega t}$$

$$=F^*(-\omega)\int f(t)e^{-i\omega t}\,dt=F^*(-\omega)F(-\omega)$$

If f(t) is real, then F(ω) is even, and $FT_\tau\left\{f_{AC}(t)\right\}=\left|F(\omega)\right|^2$

Fourier transform spectrometer

- Measure interference, subtract DC, FT to get spectrum
  - Single detector, better signal/noise

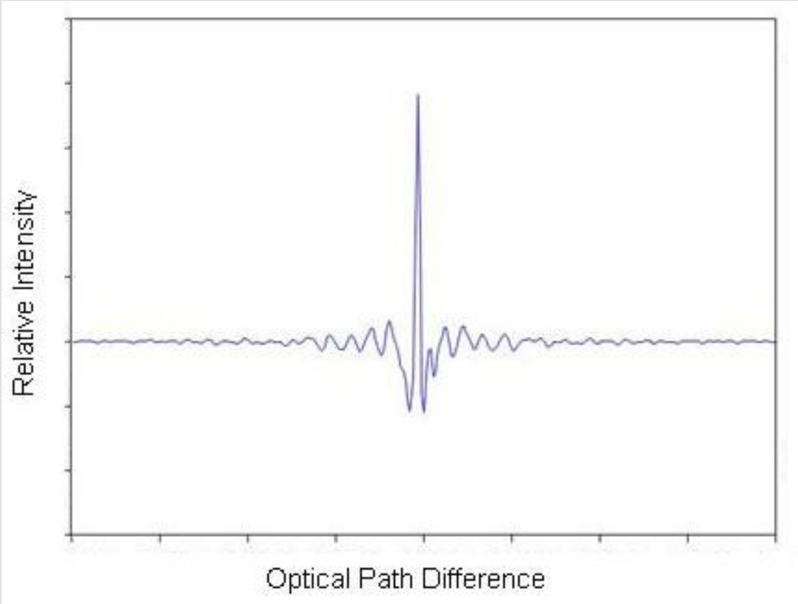


http://chemwiki.ucdavis.edu/Physical_Chemistry/Spectroscopy/Vibrational_Spectroscopy/Infrared_Spectroscopy/How_an_FTIR_Spectrometer_Operates

Coherence time

- Note that for large time delay, time averaged signal is constant (sum of two intensities)
- Beyond "coherence time" no interference
- Coherence time is inverse of spectral bandwidth

$$T_c \equiv 1/\Delta\nu$$



Intensity autocorrelation

- Second harmonic generation leads to a signal that is dependent on the square of the intensity

$$E_2 \propto \int I_1^{\,2}(t)\,dt$$

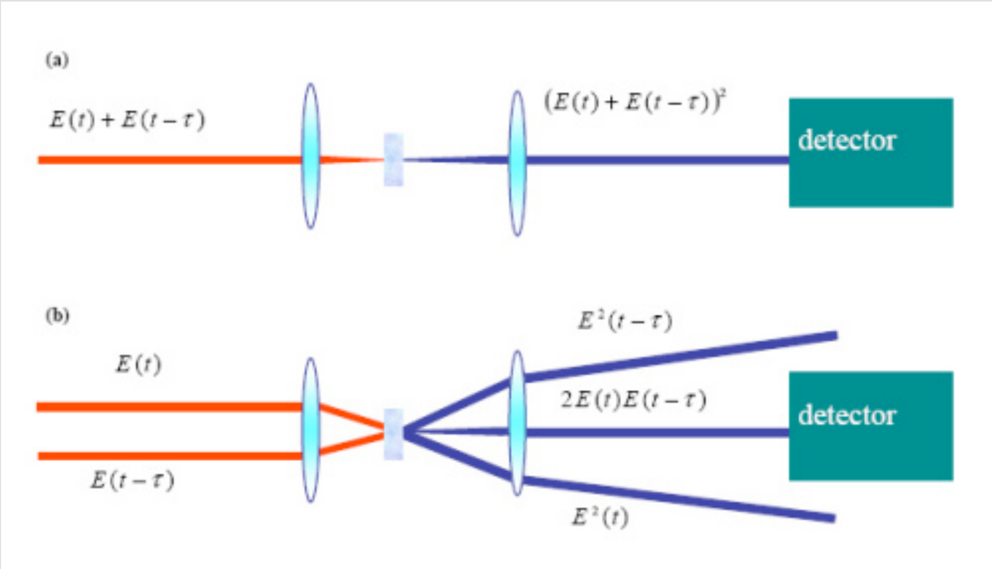
Detector integrates fast signal

- Sum-frequency mixing signal is the product of the intensity of two beams
  - Collinear

$$E_2(\tau) \propto \int \left| E_{1a}(t) + E_{1b}(t-\tau) \right|^2 dt$$

  - Non-collinear

$$E_2(\tau) \propto \int I_{1a}(t) I_{1b}(t-\tau)\,dt$$



Collinear autocorrelation

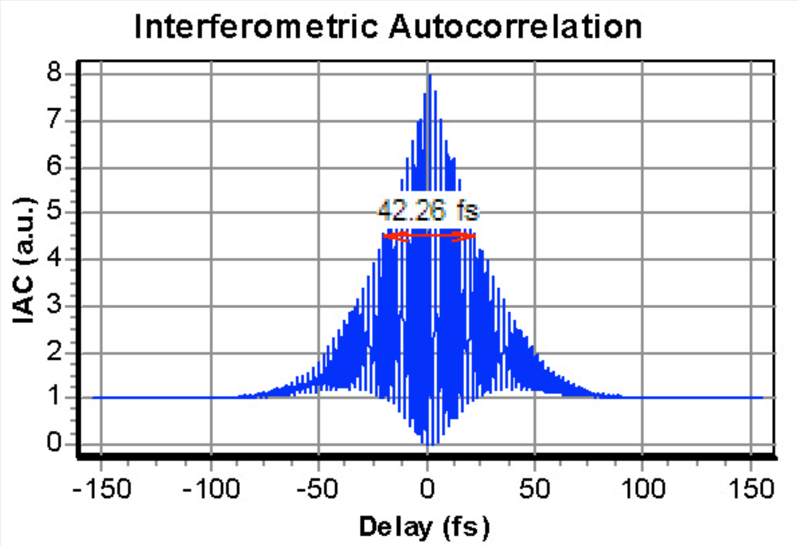
- Mixing signal interferes with doubling signal

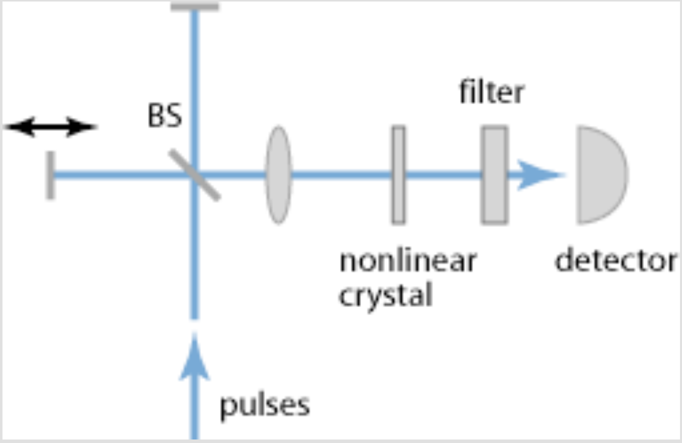
$$E_2(\tau) \propto \int \left| E_{1a}(t) + E_{1b}(t-\tau) \right|^4 dt$$

$$E_2(\tau) \propto \int \left(\left| E_{1a}(t) \right|^2 + \left| E_{1b}(t-\tau) \right|^2 + E_{1a}{}^*(t) E_{1b}(t-\tau) + E_{1a}(t) E_{1b}{}^*(t-\tau) \right)^2 dt$$

Doubling terms   Mixing terms

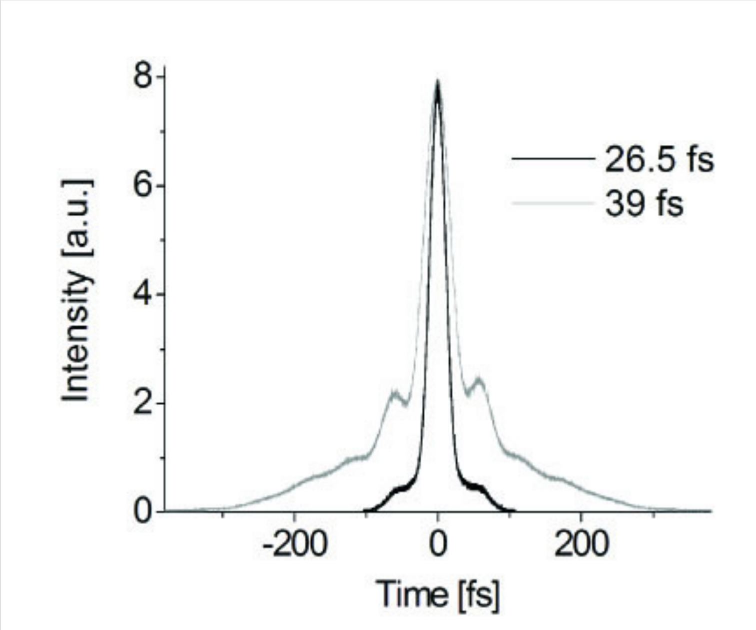
  - Interference between terms leads to an 8:1 ratio between peak and SH background

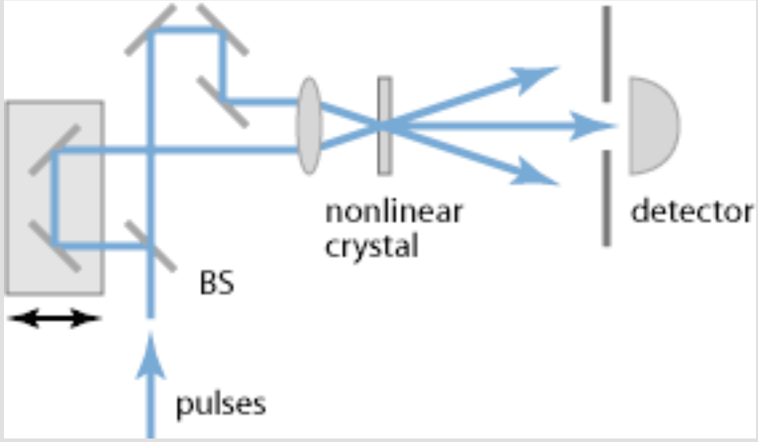




Non-collinear autocorrelation

- By crossing beams, mixing signal travels between two beams
  - Output is background free
  - Some indication of pulse structure, but not a complete characterization



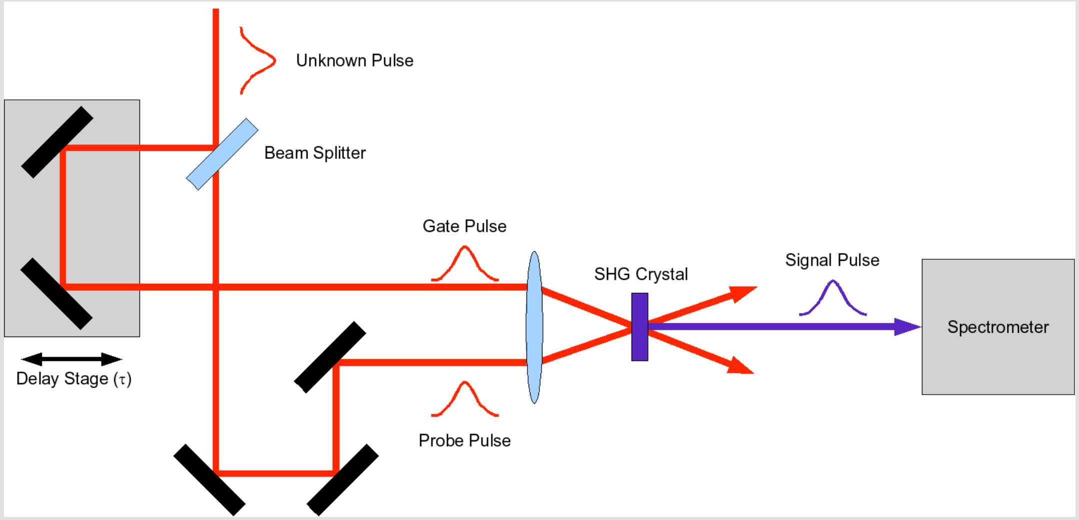


Third-order autocorrelation is also possible:

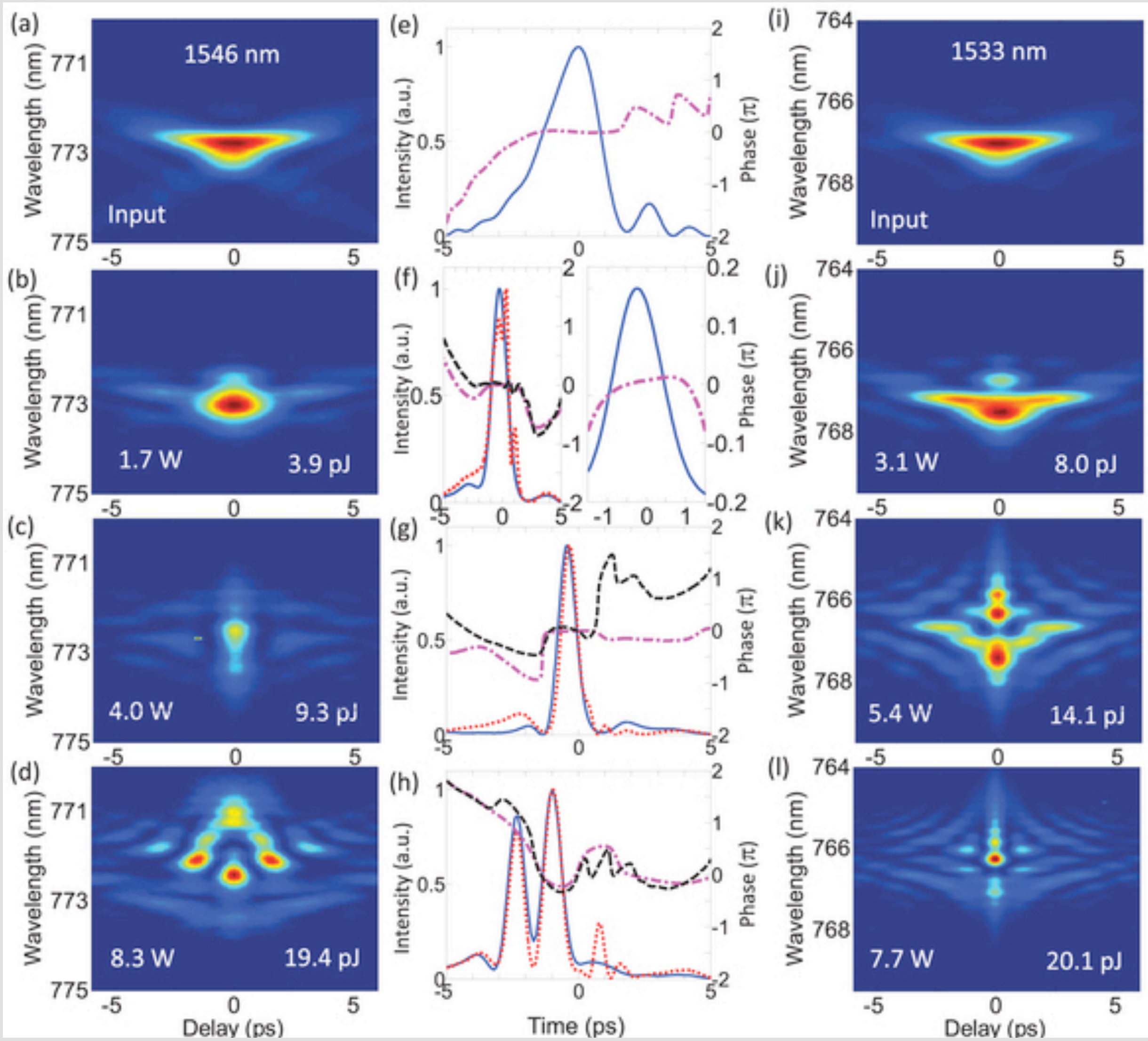
- Mix SH with fundamental
- Used for measuring ASE background

Frequency-resolved optical gating (FROG)

- Measure the spectrum of the mixing signal for each delay
  - Algorithm to retrieve amplitude and phase of field
    - Guess field, generate FROG trace
    - Use measured FROG trace to produce new guess, iterate
  - Second-order FROG is symmetric in delay
    - Can't tell which pulse is first – can't tell ordering of time in output trace, or sign of spectral phase.
    - Use a second measuremnet (e.g. add known extra chirp)



Sample FROG traces and deconvolved pulses



Soliton dynamics in the multiphoton plasma regime: *Scientific Reports* **3**, 1100 (2013)

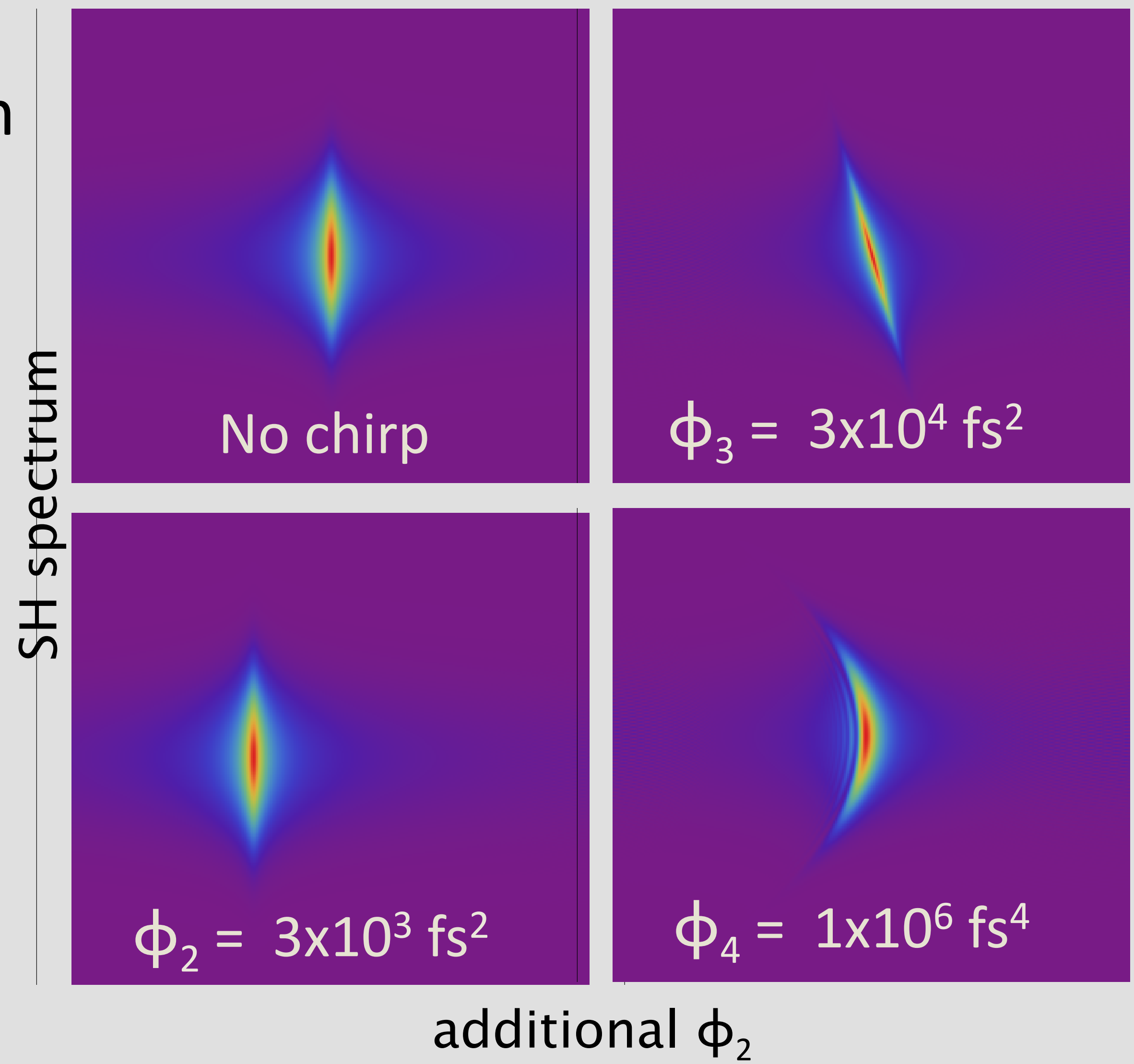
Dispersion scan

- An alternative to FROG uses a single pulse, with control over dispersion added to the pulse
  - Sinusoidal phase: MIIPS
  - Prism or grating compressor phase
- Easiest way to understand trace is if the external phase is pure second-order:
  - If the pulse has only 2nd order phase, then we can measure how much there is by seeing how much ϕ_2 must be added to compress the pulse
  - In general, the peak of the SH is at the frequency where

$$\frac{d^2\varphi}{d\omega^2} + \phi_2 = 0$$

Pure 2nd order phase dispersion scan (D-scan) pulse characterization

- Measure second harmonic spectrum as function of added ϕ_2
- Directly measure $d^2\phi/d\omega^2$ of pulse



Quasi-monochromatic fields

- Earlier we had worked with single-frequency fields, for example:
$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(k_z z - \omega t)$$
- Now we want to work with field with a more general temporal shape.
  - Assume linear polarization, plane waves in z-direction
- Separate *P* into linear and nonlinear components:
$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_L}{\partial t^2} + \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} \qquad D = \varepsilon_0 E + P_L$$
- Group linear terms together
$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 D}{\partial t^2} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 P_{NL}}{\partial t^2} \qquad \frac{1}{\varepsilon_0 \mu_0} = c^2$$

Wave equation in frequency space

- Represent all signals in ω space:

$$E(z,t)=\frac{1}{2\pi}\int E(z,\omega)e^{-i\omega t}\,d\omega \qquad P(z,t)=\frac{1}{2\pi}\int P(z,\omega)e^{-i\omega t}\,d\omega$$

$$D(z,t)=\frac{1}{2\pi}\int D(z,\omega)e^{-i\omega t}\,d\omega$$

- Now we can connect *D* and *E*: $D(z,\omega)=\varepsilon_0\varepsilon(\omega)E(z,\omega)$

- Put these expressions into the WE, do time derivatives inside integral: $\frac{\partial^2}{\partial t^2}D(z,t)=\frac{1}{2\pi}\int D(z,\omega)\left(\frac{\partial^2}{\partial t^2}e^{-i\omega t}\right)d\omega=-\frac{\omega^2}{2\pi}\int D(z,\omega)e^{-i\omega t}d\omega$

$$\frac{\partial^2}{\partial z^2}E(z,\omega)+\varepsilon(\omega)\frac{\omega^2}{c^2}E(z,\omega)=-\frac{\omega^2}{\varepsilon_0 c^2}P_{NL}(z,\omega)$$

- Now work to get back into time domain. $k^2(\omega)=\varepsilon(\omega)\frac{\omega^2}{c^2}$

Field with slowly varying envelope

- We went to ω space to be able to easily include dispersion

$$\frac{\partial^2}{\partial z^2} E(z,\omega) + k^2(\omega) E(z,\omega) = -\frac{\omega^2}{\varepsilon_0 c^2} P_{NL}(z,\omega)$$

- Represent field in terms of a slowly-varying amplitude

$$E(z,t) = A(z,t)\left(e^{i(k_0 z - \omega_0 t)} + c.c.\right) \qquad A(z,t) = \frac{1}{2\pi}\int A(z,\omega) e^{-i\omega t}\, d\omega$$

  - By shift theorem:

$$E(z,\omega) = A(z,\omega - \omega_0) e^{ik_0 z}$$

- Put this into the wave equation:

$$\frac{\partial^2}{\partial z^2}\left(Ae^{ik_0 z}\right) + k^2 Ae^{ik_0 z} = \left(\frac{\partial^2 A}{\partial z^2} + 2ik_0 \frac{\partial A}{\partial z} - k_0^2 A + k^2 A\right) e^{ik_0 z}$$

$$\frac{\partial^2 A}{\partial z^2} + 2ik_0 \frac{\partial A}{\partial z} + \left(k^2 - k_0^2\right) A = -\frac{\omega^2}{\varepsilon_0 c^2} P_{NL}(z,\omega) e^{-ik_0 z}$$

Taylor expansion of dispersion

- Do a Taylor expansion for $k(\omega)$:

$$k(\omega) = k_0 + (\omega - \omega_0)k_1 + \frac{1}{2}(\omega - \omega_0)^2 k_2 + D$$

  - D includes all high-order dispersion, will ignore for now

$$D = \sum_{n=2}^{\infty} \frac{1}{n!}(\omega - \omega_0)^n k_n$$

  - Expand k squared, drop terms higher-order than 2

$$k^2(\omega) = k_0^{\ 2} + 2k_0k_1 \cdot (\omega - \omega_0) + k_1^{\ 2} \cdot (\omega - \omega_0)^2 + 2k_0 \frac{1}{2}(\omega - \omega_0)^2 k_2$$

$$+2k_1 \cdot (\omega - \omega_0)\frac{1}{2}(\omega - \omega_0)^2 k_2 + \left(\frac{1}{2}(\omega - \omega_0)^2 k_2\right)^2 \longrightarrow \text{Small, ignore}$$

$$\frac{\partial^2 A}{\partial z^2} + 2ik_0 \frac{\partial A}{\partial z} + \left(k^2 - k_0^2\right)A = -\frac{\omega^2}{\varepsilon_0 c^2} P_{NL}(z,\omega) e^{-ik_0 z}$$

$$\frac{\partial^2 A}{\partial z^2} + 2ik_0 \frac{\partial A}{\partial z} + \left(2k_0k_1 \cdot (\omega - \omega_0) + \left(k_1^{\ 2} + k_0k_2\right) \cdot (\omega - \omega_0)^2\right)A = -\frac{\omega^2}{\varepsilon_0 c^2} P_{NL} e^{-ik_0 z}$$

Transform back to time domain

$$\frac{\partial^2 A}{\partial z^2} + 2ik_0 \frac{\partial A}{\partial z} + \left(2k_0 k_1 \cdot (\omega - \omega_0) + \left(k_1^{\ 2} + k_0 k_2\right) \cdot (\omega - \omega_0)^2\right) A = -\frac{\omega^2}{\varepsilon_0 c^2} P_{NL}(z,\omega) e^{-ik_0 z}$$

- Now inverse FT to go back to time domain
  - Multiply by $e^{-i(\omega-\omega_0)t}$, integrate
  - Derivative theorem:

Integrate by parts

$$FT\left\{\frac{\partial}{\partial t} f(t)\right\} = \int \frac{\partial f}{\partial t} e^{i\omega t} dt \qquad \rightarrow f e^{i\omega t}\Big|_{-\infty}^{\infty} - i\omega \int f\, e^{i\omega t} dt = -i\omega F(\omega)$$

Zero if $\lim_{t\to\pm\infty} f(t) = 0$

$$FT^{-1}\left\{(\omega - \omega_0)^n A(\omega - \omega_0)\right\} = \left(i\frac{\partial}{\partial t}\right)^n \tilde{A}(t)$$

$$\frac{\partial^2 \tilde{A}}{\partial z^2} + 2ik_0 \frac{\partial \tilde{A}}{\partial z} + \left(2ik_0 k_1 \frac{\partial}{\partial t} - \left(k_1^{\ 2} + k_0 k_2\right)\frac{\partial^2}{\partial t^2}\right)\tilde{A} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 \tilde{P}_{NL}}{\partial t^2} e^{-i\left(k_0 z - \omega_0 t\right)}$$

NL polarization in the time domain

- In the time domain, write PNL in terms of an envelope and carrier:

$$\tilde{P}_{NL}(z,t) = \tilde{p}(z,t)\,e^{i(k_0 z - \omega_0 t)}$$

$$\frac{\partial \tilde{P}_{NL}}{\partial t} = \left(-i\omega_0 \tilde{p} + \frac{\partial \tilde{p}}{\partial t}\right) e^{i(k_0 z - \omega_0 t)} = -i\omega_0 \left(1 + \frac{i}{\omega_0}\frac{\partial}{\partial t}\right) \tilde{p}\, e^{i(k_0 z - \omega_0 t)}$$

$$\rightarrow \frac{\partial^2 \tilde{P}_{NL}}{\partial t^2} = -\omega_0^{\;2} \left(1 + \frac{i}{\omega_0}\frac{\partial}{\partial t}\right)^2 \tilde{p}\, e^{i(k_0 z - \omega_0 t)} \approx -\omega_0^{\;2}\, \tilde{p}\, e^{i(k_0 z - \omega_0 t)}$$

Slowly varying envelope

$$\frac{\partial^2 \tilde{A}}{\partial z^2} + 2ik_0 \frac{\partial \tilde{A}}{\partial z} + \left(2ik_0 k_1 \frac{\partial}{\partial t} - \left(k_1^{\;2} + k_0 k_2\right)\frac{\partial^2}{\partial t^2}\right)\tilde{A} = -\frac{\omega_0^{\;2}}{\varepsilon_0 c^2}\,\tilde{p}$$

Refererence frame moving at group velocity

- Re-group terms:

$$\left[\frac{\partial^2}{\partial z^2}+2ik_0\left(\frac{\partial}{\partial z}+k_1\frac{\partial}{\partial t}\right)-\left(k_1{}^2+k_0k_2\right)\frac{\partial^2}{\partial t^2}\right]\tilde{A}=-\frac{\omega_0{}^2}{\varepsilon_0c^2}\tilde{p}$$

  – Define moving coordinates $z'=z$ $\quad\tau=t-k_1z\quad$ $\frac{\partial}{\partial t}=\frac{\partial}{\partial\tau}$

$$\frac{\partial}{\partial z}=\frac{\partial z'}{\partial z}\frac{\partial}{\partial z'}+\frac{\partial\tau}{\partial z}\frac{\partial}{\partial\tau}=\frac{\partial}{\partial z'}-k_1\frac{\partial}{\partial\tau}$$

$$\frac{\partial^2}{\partial z^2}=\left(\frac{\partial}{\partial z'}-k_1\frac{\partial}{\partial\tau}\right)^2=\frac{\partial^2}{\partial z'^2}-2k_1\frac{\partial}{\partial z'}\frac{\partial}{\partial\tau}+k_1{}^2\frac{\partial^2}{\partial\tau^2}$$

$$\left[\frac{\partial^2}{\partial z'^2}-2k_1\frac{\partial}{\partial z'}\frac{\partial}{\partial\tau}+\cancel{k_1{}^2\frac{\partial^2}{\partial\tau^2}}+2ik_0\left(\frac{\partial}{\partial z'}-\cancel{k_1\frac{\partial}{\partial\tau}}+\cancel{k_1\frac{\partial}{\partial t}}\right)-\left(\cancel{k_1{}^2}+k_0k_2\right)\frac{\partial^2}{\partial t^2}\right]\tilde{A}=-\frac{\omega_0{}^2}{\varepsilon_0c^2}\tilde{p}$$

$$\left[-2k_1\frac{\partial}{\partial z'}\frac{\partial}{\partial\tau}+2ik_0\frac{\partial}{\partial z'}-k_0k_2\frac{\partial^2}{\partial t^2}\right]\tilde{A}=-\frac{\omega_0{}^2}{\varepsilon_0c^2}\tilde{p}$$

Drop 2nd derivative
Slowly-varying envelope approx

Simplify to the NLS

- Group terms

$$\left[-2k_1 \frac{\partial}{\partial z'}\frac{\partial}{\partial \tau} + 2ik_0 \frac{\partial}{\partial z'} - k_0 k_2 \frac{\partial^2}{\partial t^2}\right]\tilde{A} = \left[2ik_0 \frac{\partial}{\partial z'}\left(1 + i\frac{k_1}{k_0}\frac{\partial}{\partial \tau}\right) - k_0 k_2 \frac{\partial^2}{\partial t^2}\right]\tilde{A} = -\frac{\omega_0{}^2}{\varepsilon_0 c^2}\tilde{p}$$

$$\frac{k_1}{k_0} = \frac{dk/d\omega|_{\omega_0}}{n\omega_0/c} = \frac{1}{\omega_0}\frac{\mathrm{v}_{ph}}{\mathrm{v}_g} \approx \frac{1}{\omega_0}, \quad \text{so } 1 + i\frac{k_1}{k_0}\frac{\partial}{\partial \tau} \approx 1 + \frac{i}{\omega_0}\frac{\partial}{\partial \tau}$$

Again, by SVEA, drop second term

– Define NL polarization envelope in terms of intensity

$$\tilde{p} = 2\varepsilon_0 n_0 n_2 I\,\tilde{A}$$

$$\left[2ik_0 \frac{\partial}{\partial z'} - k_0 k_2 \frac{\partial^2}{\partial t^2}\right]\tilde{A} = -\frac{\omega_0{}^2}{c^2}2n_0 n_2 I\,\tilde{A}$$

$$k_0 = \frac{\omega_0}{c}n_0$$

$$\boxed{\frac{\partial}{\partial z'}\tilde{A} = -i\frac{1}{2}k_2\frac{\partial^2 \tilde{A}}{\partial t^2} + i\frac{\omega_0}{c}n_2 I\,\tilde{A}}$$

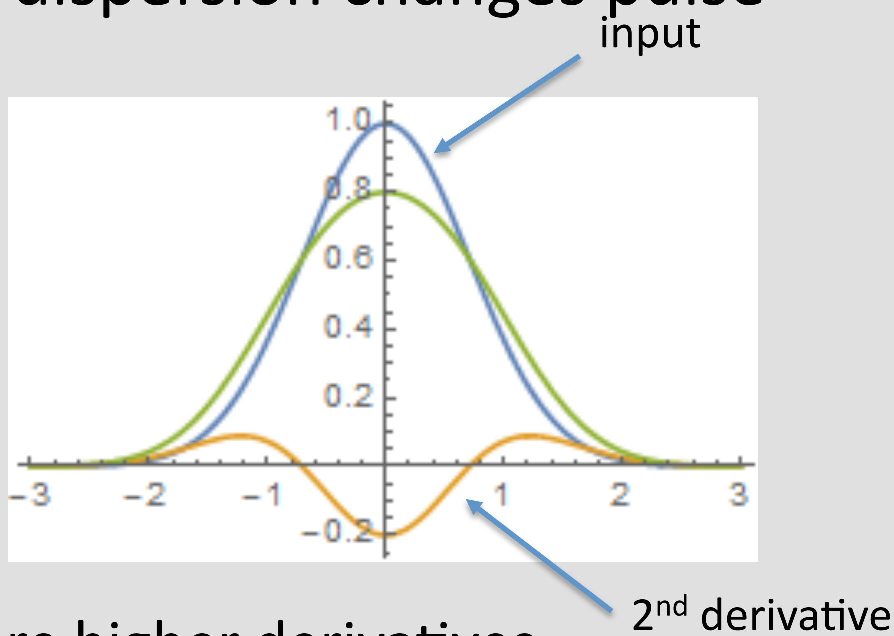
This is called the nonlinear Schroedinger eqn NLS

NLS derivation: observations

- w/o nonlinearity, 2nd order dispersion changes pulse duration

$$\frac{\partial}{\partial z'}\tilde{A} = -i\frac{1}{2}k_2\frac{\partial^2 \tilde{A}}{\partial t^2}$$

$$\tilde{A}_{j+1} = \tilde{A}_{j+1} + \Delta z\left(-i\frac{1}{2}k_2\frac{\partial^2 \tilde{A}}{\partial t^2}\right)$$



  - higher-order phase terms are higher derivatives
- w/o dispersion, pulse experiences SPM
  - Spectrum is broadened

Fourier split-step scheme for numerical solutions

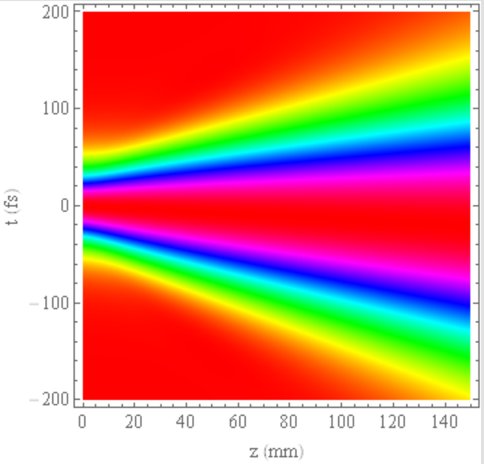
- Rough outline
  - Set up grids, input pulse
  - Represent pulse in frequency domain, advance one step in z to apply dispersion
  - Inverse FT to time domain, advance same step in z to apply nonlinear phase shift
  - FT back to frequency – repeat
- Advanced variations
  - Stagger steps (½ step for dispersion, full for NL, ½ dispersion)
  - Since dispersion is evaluated in freq space, can use high-order w/o approximation

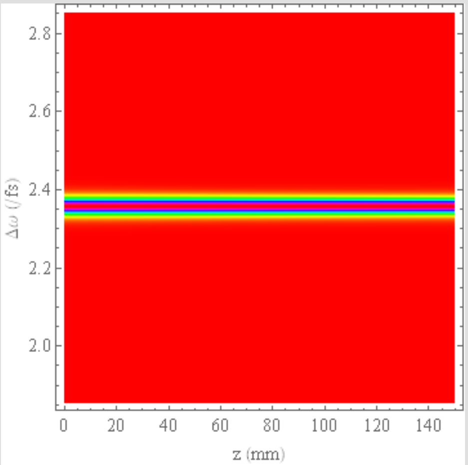
Applications of split step method

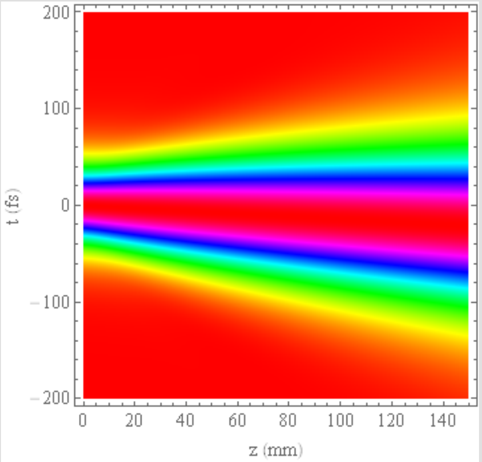
- NL propagation of pulses in fibers
  - No spatial variation for guided mode
  - Describes soliton dynamics, pulse compression
- Apply to other domains:
  - QM evolution of particle wavepackets
  - Spatial propagation
  - Mixed spatio-temporal propagation

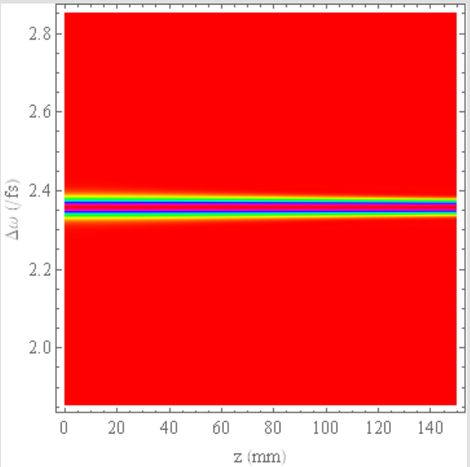
NL propagation examples

- Low intensity:



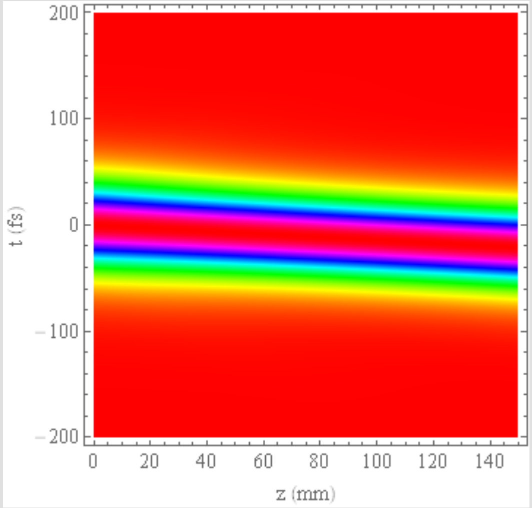


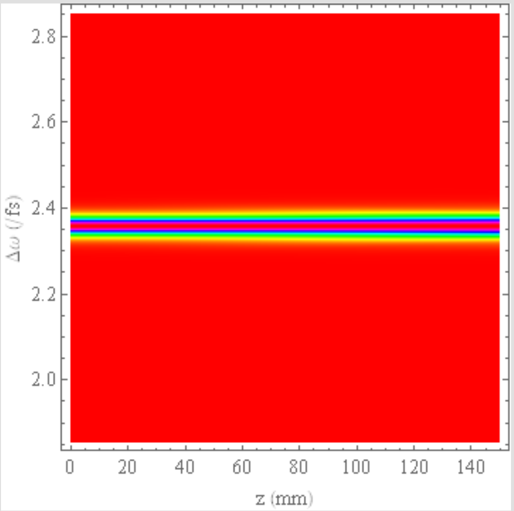




NL propagation examples: solitons

- N = 1





- N=2