

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points) Conceptual Questions

- (a) Suppose the f is defined on a finite domain of \mathbb{R} and is such that $f(-x) = -f(x)$. Does f have a Fourier series representation? If so, then does it have cosines in it? Does f have a Fourier transform?

$$f(-x) = -f(x) \Rightarrow f \text{ is odd}$$

finite domain $\xrightarrow{*} F.S. \text{ Rep., odd} \Rightarrow \text{no cosines}$

finite domain $\xrightarrow{**} F.T. \text{ Rep.}$

$\xrightarrow{*}$ By periodic Extension $\xrightarrow{**}$ Extend the def. to Rest of \mathbb{R}

- (b) What is the relationship/connection between Fourier integrals and Fourier series? What is the purpose of each?

A F.S. is a Rep. for a periodic f_{per} in the sine/cosine basis which uses a countable linear combination. A Fourier Integral is found by taking $L \rightarrow \infty$ in a F.S. and can Rep. nonperiodic f_{nonper} using an integral over a uncountable lin. comb. of basis f_{per} .

- (c) Suppose the f has an even symmetry then does its Fourier transform have a symmetry? If so then what is it?

$$f \text{ is even/odd} \Rightarrow \hat{f} \text{ is even/odd}$$

2. (10 Points) Given the following,

$$\frac{2 \sin(n\pi)}{n} = \int_{-\pi}^{\pi} f(x) \cos(nx) dx, \quad 0 = \int_{-\pi}^{\pi} f(x) dx, \quad \frac{2 \cos(n\pi)}{n} = \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

$$10 = \int_{-\pi}^{\pi} g(x) dx, \quad \frac{2(e^{in\pi} - e^{-in\pi})}{n^2} = \int_{-\pi}^{\pi} g(x) e^{-inx} dx$$

where n is an integer:

(a) Find the real Fourier series of f . Is f even or odd?

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) =$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^n}{n} \sin(nx) \Rightarrow f \text{ is odd}$$

(b) Find the complex Fourier series of g . Is g even or odd?

$$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 = \frac{10}{2\pi} \Rightarrow g \text{ is even.}$$

3. (10 Points) Given that the coefficients of a complex Fourier series are $c_0 = 0$ and $c_n = \frac{2(-1)^n}{n^2}$, find the real Fourier series.

$$\text{Let } f(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} c_n e^{inx} = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \left(\cos\left(\frac{n\pi}{L}x\right) - i \sin\left(\frac{n\pi}{L}x\right) \right) +$$

$$+ \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \left(\cos\left(\frac{n\pi}{L}x\right) + i \sin\left(\frac{n\pi}{L}x\right) \right) = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos\left(\frac{n\pi}{L}x\right)$$

4. (10 Points) Given,

$$f(x) = A, \quad x \in (0, a), \quad (1)$$

calculate the Fourier cosine and Fourier sine half-range expansions of f .

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{a}x\right) + b_n \sin\left(\frac{n\pi}{a}x\right).$$

$$b_n = \frac{2}{a} \int_0^a A \sin\left(\frac{n\pi}{a}x\right) dx = \frac{2A}{n\pi} \left[-\cos\left(\frac{n\pi}{a}x\right) \right]_0^a = \frac{-2A}{n\pi} [(-1)^n - 1]$$

Cosine: $a_0 = A, a_n = 0$

$$\text{for } f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{a}x\right) = A$$

5. (10 Points) Given,

$$f(x) = \begin{cases} -x, & -1 < x < 0 \\ x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}, \quad (2)$$

calculate $\hat{f}(\omega)$.

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx = \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^0 -x e^{i\omega x} dx + \int_0^1 x e^{i\omega x} dx \right] = \frac{1}{\sqrt{2\pi}} \left[\int_0^1 -x e^{-i\omega x} dx + \int_0^1 x e^{i\omega x} dx \right] = \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_0^1 x \right] e^{i\omega x} \Big|_0^1 - \frac{1}{\sqrt{2\pi}} \left[\int_0^1 x \right] e^{-i\omega x} \Big|_0^1 = \\ &\quad \text{I'm a dork.} \\ &\quad f \text{ is even} \Rightarrow \hat{f}(\omega) = \frac{2}{\sqrt{2\pi}} \int_0^1 x \cos(\omega x) dx = \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{x \sin(\omega x)}{\omega} \Big|_0^1 + \frac{1}{\omega^2} \cos(\omega x) \Big|_0^1 \right] = \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{\sin(\omega)}{\omega} + \frac{\cos(\omega)}{\omega^2} - \frac{1}{\omega^2} \right] \end{aligned}$$