1.5 Existence and Uniqueness Theorems

Existence Theorem: Suppose $f(t,y)$ is a continuous function in a rectangle of the form $\{(t,y)|a<t<b, c<y<d\}$ in the ty-plane. If (t_0, y_0) is a point in this rectangle, then there exists an $\epsilon > 0$ and a function $y(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ that solves the initial-value problem:

$$\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0.$$

Uniqueness Theorem: Suppose $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous functions in a rectangle of the form $\{(t,y)|a<t<b, c<y<d\}$ in the ty-plane. If (t_0, y_0) is a point in this rectangle and if $y_1(t)$ and $y_2(t)$ are two functions that solve the initial-value problem:

$$\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0$$

for all t in the interval $t_0 - \epsilon < t < t_0 + \epsilon$ (ϵ is a positive number), then

$$y_1(t) = y_2(t)$$

for $t_0 - \epsilon < t < t_0 + \epsilon$. That is the solution to the initial-value problem is unique.