MATH 348 - Advanced Engineering Mathematics March 3, 2009

Homework 5, Spring 2009 **Due**: February 27 , 2009

Integration Review

1. Calculate the following integrals.
   (a) $\int x^3 \cos(5x)dx$
   (b) $\int x^2 \sin(2x^3)dx$
   (c) $\int e^{ax} \cos(bx)dx$
   (d) $\int_0^{2\pi} \sin(nx)\cos(mx)dx$, where $n, m \in \mathbb{Z}$

representation of vectors in vector spaces

2. Consider the vector space $\mathbb{R}^2$, which is the space of all vectors of the form $\hat{\mathbf{v}} = (x, y)$. We know that $\hat{\mathbf{i}} = (1, 0)$ and $\hat{\mathbf{j}} = (0, 1)$ forms the standard basis for $\mathbb{R}^2$. That is, any vector in the space can be created as a linear combination, $\mathbf{v} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}}$, of these vectors. However, it is possible to choose other basis vectors and still represent all vectors in the space. Consider defining:

$$\hat{\mathbf{i}} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \quad \hat{\mathbf{j}} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \\ \frac{\sqrt{2}}{2} \end{bmatrix}. \tag{1}$$

The previous vectors form another orthonormal-basis for $\mathbb{R}^2$.

   (a) Show that the vectors are orthonormal by verifying the inner-products $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0$ and $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = 1$.
   (b) Show that any vector for $\mathbb{R}^2$ can be created as a linear combination of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$. That is, given,

$$\hat{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}}, \tag{2}$$

   show that c_1, c_2, can be found in terms of x_1 and x_2.

Hint: Recall that the product $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}}$ is called an inner-product or dot-product and is defined to be $\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{v}}_2 = (x_1, y_1) \cdot (x_2, y_2) = x_1x_2 + y_1y_2$. For part (b) you may want to consider taking inner-products on both sides of the equation.

3. In the previous problem the vector space is $\mathbb{R}^2$. However, there are many other mathematical objects that when grouped together satisfy the same rules as the vectors in $\mathbb{R}^2$. One such grouping is the set of all 2π-periodic functions. We call the space of all 2π-periodic functions an abstract vector space. Each 'vector' in this space is a function, f, which has the property that $f(x+2\pi) = f(x)$. Our goal is to represent any vector in the space by a linear combination of a set of standard basis vectors. Our hope is that there exists an orthonormal basis for this space so that the coefficients in the sum are easy to calculate. Before we do this we must define an inner product. Choose:

$$f(x) \cdot g(x) = \int_{-\pi}^{\pi} f(x)g(x)dx. \tag{3}$$

Let, $f(x) = \cos(x)$ and $g(x) = \sin(x)$ and show the following orthogonality relations:

   (a) $f(nx) \cdot f(mx) = \pi\delta_{nm}$, where $n, m \in \mathbb{Z}$.
   (b) $g(nx) \cdot g(mx) = \pi\delta_{nm}$, where $n, m \in \mathbb{Z}$.

(c) $f(nx) \cdot g(mx) = 0, \quad n.m \in \mathbb{Z}$, no matter the choice of n and m.

Hint: Here the function δ_{nm} is called the Kronecker delta function and is defined to be zero if $n \neq m$ and one if $n = m$.
Comment: These relationships are important and is a step in showing that the following collection of 'vectors'

$$\left\{ \frac{1}{\sqrt{\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, \frac{\cos(3x)}{\sqrt{\pi}}, \frac{\sin(3x)}{\sqrt{\pi}}, \ldots \right\} \tag{4}$$

forms an orthonormal basis for the space of all 2π-periodic functions.

Fourier Series - Introduction

4. Let $f(x) = x^2$, $(-\pi < x < \pi)$ be a 2π-periodic function.

   (a) Sketch a graph f on $[-4\pi, 4\pi]$.

   (b) Go to `http://www.tutor-homework.com/grapher.html` and graph the following functions,

$$\begin{aligned} f_1(x) &= \frac{\pi^2}{3} - 4\left(\cos(x) - \frac{\cos(2x)}{4} + \frac{\cos(3x)}{9}\right) \\ f_2(x) &= x^2 \end{aligned}$$

   (c) Print your results and comment on your three graphs.

5. Readings

   (a) Go to `http://en.wikipedia.org/wiki/Fourier_series` and read the introductory material on Fourier Series and describe in your own words the purpose and application of Fourier Series.

   (b) Using the Java Applet found at,
   `http://www.sunsite.ubc.ca/LivingMathematics/V001N01/UBCExamples/Fourier/fourier.html`,
   use the applet to graph a truncated Fourier Series approximating the saw-tooth function. What occurs at the points jump-discontinuity?

   (c) Read, as much as you can, of `http://en.wikipedia.org/wiki/Gibbs_phenomenon`. The sum of a finite, or infinite amount of periodic functions is periodic. Is this always true for both finite and infinite sums of continuous functions? Can you think of a counterexample? [1]

[1] These questions are meant to lead you. Remembering that sine and cosine are examples of continuous periodic functions, you should be thinking about the following string of thoughts.

i. Fourier series represent an 'arbitrary' periodic function in terms of known periodic functions.

ii. Increasing the number of terms in a Fourier series creates better and better sinusoidal wave-form fits of the function f and in the limit of infinitely many terms this fit is exact 'almost-everywhere'.

iii. Hopefully by the time you do this problem we would have mentioned in class that the Fourier series representation of a function converges in the sense of averages and that since jump-discontinuities are integrable-discontinuities the Fourier series would average the right and left hand limits of the function at the point of discontinuity. This will happen indifferent to the actual value of the function at the point of discontinuity. Thus the Fourier series may actually differ from its function at the boundaries of its periodic-domains! In this way we take = to mean equality *almost everywhere* (`http://en.wikipedia.org/wiki/Almost_everywhere`).

So, we have that the sawtooth example from class and the square-wave example online are examples where the infinite sum of continuous periodic functions converges to a periodic function with jump-discontinuities.