The Language of and Solutions to Linear Systems

Text: 1.3-1.5, 1.7 Section Overviews: 1.3-1.5, 1.7

Quote of Homework Two	
Out here in the fields, I fight for my meals. I get my back into my living.	
	The Who : Baba O'Riley (1971)

1. Warm Ups

1.1. **Existence and Uniqueness of Solutions and Pivot Structure.** Determine the values of h and k so that,

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array}\right],$$

(1) Is consistent (has a solution). [1]
(2) Is consistent with a unique solution. [2]
(3) Is inconsistent.

1.2. **Coefficient Data and Existence and Uniqueness of Solutions.** Assuming that $a \neq 0$, find an equation that restricts a, b, c, d so that the following system has only the trivial solution. [3]

$$ax_1 + bx_2 = 0 \tag{1}$$
$$cx_1 + dx_2 = 0. \tag{2}$$

1.3. **Language of Vector Equations.** Given,

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix},$$

and observe that the first column plus twice the second column equals the third column. Find a nontrivial solution to the associated homogeneous system. [4]

[1] A system is consistent if there are no inconsistent rows.

[2] A system is consistent with a unique solution if there are no inconsistent rows and there are as many pivots as variables.

[3] Hint: Find the echelon form of the associated matrix equation and from this echelon form find a relation/rule involving a, b, c, d so that the augmented matrix has as many pivots as variables. If you have taken the *determinant* of 2×2 matrices then you can check your work.

[4] Hint: You could solve the system $\mathbf{Ax} = \mathbf{0}$ and find ALL solutions to the homogeneous problem but you aren't asked for this. The idea here is that you could avoid row-reduction altogether. How? Well, write the system in its associated vector form and try to find the vector's coefficients.

2. The Vector Equation: $\sum_{j=1}^{n} x_j \mathbf{a}_j = \mathbf{b},\ \mathbf{a}_j, \mathbf{b} \in \mathbb{R}^m$

Given,[5]

$$\mathbf{A}_1 = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix},$$

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix},$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

2.1. **Linear Combinations.** Is $\mathbf{b}_1$ a linear combination of the columns of $\mathbf{A}_1$?

2.2. **Linear Dependence.** Determine all values for h such that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a linearly dependent set.

2.3. **Linear Independence.** Determine all values for h such that $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ forms a linearly independent set.

2.4. **Spanning Sets.** How many vectors are in $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$? How many vectors are in span(S)? Is $\mathbf{y} \in$span(S)?

2.5. **Introduction to Matrix Spaces.** Is $\mathbf{b}_2$ a solution to the homogeneous problem of $\mathbf{A}_2$? Is $\mathbf{b}_2$ a linear combination of the columns of $\mathbf{A}_2$?

3. The Matrix Equation: $\mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{A} \in \mathbb{R}^{m \times n},\ \mathbf{x} \in \mathbb{R}^n,\ \mathbf{b} \in \mathbb{R}^m$

Given,[5]

$$\mathbf{A}_1 = \begin{bmatrix} 1 & -3 & 0 \\ -1 & 1 & 5 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 6 & 18 & -4 \\ -1 & -3 & 8 \\ 5 & 15 & -9 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 3 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}, \quad \mathbf{A}_5 = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix},$$

$$\mathbf{b}_1 = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 20 \\ 4 \\ 11 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}, \quad \mathbf{b}_5 = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix}.$$

3.1. **Algebra.** Find all solutions to $\mathbf{A}_i\mathbf{x} = \mathbf{b}_i$ for $i = 1, 2, 3, 4, 5$.

3.2. **Geometry.** Describe or plot the geometry formed by the linear systems and their solution sets.

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[5] All of the following problems require you to apply row-reduction to the appropriate augmented matrix and then interpreting the results. You have already done the row-reductions in homework1, which means that all you have to do now is interpret the pivot structure.

4. Linear Independence

Given,

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 9 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}. \tag{3}$$

4.1. **Row-Reduction.** Find the row echelon form of the matrix $\mathbf{V}$ whose columns are the given vectors.

4.2. **Linear Independent Sets.** Does $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ form a linearly independent set? What about the sets $S_2 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $S_3 = \{\mathbf{v}_1, \mathbf{v}_2\}$, $S_4 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$, $S_5 = \{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

4.3. **Spanning Sets.** Does S_1 span $\mathbb{R}^3$? What about S_2, S_3, S_4, S_5? If you were going to span $\mathbb{R}^3$ then which of these sets would you choose? [6]

5. Application

Suppose you have a set S of three points in $\mathbb{R}^2$,

$$S = \{(t_1, p_1), (t_2, p_2), (t_3, p_3)\}, \tag{4}$$

which you seek to interpolate with the quadratic polynomial $p(t) = a_0 + a_1 t + a_2 t^2$.

5.1. **Interpolations and Linear Systems.** Using S and $p(t)$ define a linear system of equations in the a_0, a_1, a_2 variables.[7]

5.2. **Existence and Uniqueness in Interpolation.** Determine which of the following sets of points can be uniquely interpolated by $p(t)$. [8]

$$S_1 = \{(1, 12), (2, 15), (3, 16)\}$$
$$S_2 = \{(1, 12), (1, 15), (3, 16)\}$$
$$S_3 = \{(1, 12), (2, 15), (2, 15)\}$$

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[6]Remark: This problem is meant to demonstrate a few reoccurring points in linear algebra. The idea is that we typically work problems backwards in the sense that you start with a space of vectors, say $\mathbb{R}^n$, and ask the question,

- Given a 'vector-space' can we 'reach' every 'point' in the space.

More importantly, how can we do this with a minimal set of vectors? In this problem the vector-space is $\mathbb{R}^3$ and from calculus we know that we need only three linearly-independent vectors, typically $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, to reach every point via linear combination (aka span the space), $\mathbf{x} = x_1\hat{\mathbf{i}} + x_2\hat{\mathbf{j}} + x_3\hat{\mathbf{k}} \in \mathbb{R}^3$. Consequently, if we choose any four vectors they must be linearly-**de**pendent, S_1, which means that some vectors point in redundant directions. However, that doesn't mean that we can pick any three vectors and still span the space, S_5. So, we have to make careful choice to take enough vectors to span the space but **not so many** that the set of vectors is linearly-**de**pendent. When we have made this careful choice we have secretly constructed a coordinate system or basis for the space. This choice is not unique, S_2, S_4.

[7]Hint: This problem is meant to trick you. Clearly, $p(t)$ is a nonlinear equation in the t-variable but once you have chosen a t-value then it is a linear equation in the coefficient variables. If you choose many t-values then you have many linear equations and now the tools of linear algebra apply.

[8]Your choice! We have two ways to approach this problem. First, you have a linear system and thus row-reduction and interpretation of pivot structure. However, if you think about the geometry of the points and the possible graphs of quadratic polynomials you should be able to determine, which of the points can be interpolated.