MATH332 - July 27, 2009 **NAME**:

Exam II - 50 Points

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points) True/False and Short Response

   (a) Mark each statement True or False. No justification is needed.

   i. The set of all $n-$degree polynomials, $p(t)$, such that $p(2) = 3$ forms a vector subspace of $\mathbb{P}_n$.

   ii. Given $\mathbf{A}_{m\times n}$ the null space of $\mathbf{A}$ is a subspace of $\mathbb{R}^m$.

   iii. The column space of $\mathbf{A}_{m\times n}$ is the set of all solutions to $\mathbf{Ax} = \mathbf{b}$.

   iv. The columns of an $n \times n$ invertible matrix forms a basis for $\mathbb{R}^n$.

   v. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are two linearly independent eigenvectors then they must correspond to two different eigenvalues.

   (b) Please respond to **both** of the following. Provide justification for your position.

   i. Could a 6×9 matrix have a two-dimensional null space?

   ii. Suppose that $\mathbf{P}_{\mathfrak{B}}$ is a change of coordinates matrix. List three properties of $\mathbf{P}_{\mathfrak{B}}$.

2. (10 Points) Quickies:

(a) Determine the dimension of the vector space formed by the span of each of the three sets.

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad S_3 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

(b) Given,

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}. \tag{1}$$

Find one eigenvalue of $\mathbf{A}$.

(c) Given,

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

what is the dimension of the null-space, column-space and row-space of $\mathbf{A}$?

(d) Given,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}, \tag{3}$$

find one eigenvalue of $\mathbf{A}$.

3. (10 Points) Proofs:

   (a) Show that if two matrices $\mathbf{A}$ and $\mathbf{B}$, are similar, $\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1}$, then they have the same eigenvalues.

   (b) If the set of all $n \times n$ matrices forms a vector space then show that the subset defined by all $n \times n$ symmetric matrices, $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$, forms a vector subspace.

4. (10 Points) Given,

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}. \tag{4}$$

   Find a basis for the null-space, column-space and row-space of $\mathbf{A}$.

5. (10 Points) Given,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}. \tag{5}$$

Using diagonalization calculate $\mathbf{A}^4$.

6. (Extra Credit) Please choose **one** of the following:

(a) Find a basis for the set of all vectors of the form $\begin{bmatrix} a - 2b + 5c \\ 2a + 5b - 8c \\ -a - 4b + 7c \\ 3a + b + c \end{bmatrix}$, where $a, b, c \in \mathbb{R}$.

(b) Suppose that $\mathbf{A}_{n\times n}$ has $n-$distinct eigenvalues such that $|\lambda_i| < 1$ for $i = 1, 2, 3, \ldots, n$. Show that $\lim_{k\to\infty} \mathbf{A}^k$ is the zero matrix.