

Antennas - radiation from collections of charges

2 assumptions:

- 1) localized charge/current distribution (size d)
- 2) source oscillates w/ frequency ω ($\lambda = 2\pi c/\omega$)

Observation distance - r

Must make approximations to do calculations: order these dimensions
 λ, d, r

Scaling analysis:

$$\vec{E} = e \left[\frac{(\hat{R} - \beta)(1 - \beta^2)}{K^3 R^2} + \frac{\hat{R} \times ((\hat{R} - \beta) \times \vec{a})}{c^2 K^3 R} \right]$$

$$\vec{B} = \hat{R} \times \vec{E}, \quad \vec{S} = \frac{c}{4\pi} (\vec{E} \times \vec{H})$$

compare relative magnitude of terms.

K^3 common to all

single charge: oscillates over dist d at freq. $\nu =$

$$u \sim d \cdot \nu \quad \beta \sim d \nu / c \sim d / \lambda \quad a/c^2 \sim d / \lambda^2$$

since $\beta < 1$, single charge must have $d < \lambda$

example: $\nu \sim 1 \text{ MHz}$ (AM radio)

$$\rightarrow \lambda \sim 300 \text{ m}$$

$$d \sim 1 \text{ cm}$$

$$u \sim d \nu \sim 10^{-2} \text{ m} \cdot 10^6 \text{ s}^{-1} \sim 10^4 \text{ m/s}$$

$$\rightarrow \beta \sim d/\lambda \sim 3 \times 10^{-5}$$

$$\left. \begin{array}{l} 100 \text{ MHz FM} \\ 3 \text{ m} \end{array} \right\}$$

$$\left. \begin{array}{l} 10^8 \text{ m/s} \\ 3 \times 10^{-3} \end{array} \right\}$$

$$E \sim \frac{1}{r^2} + \frac{a}{c^2 R} \sim \frac{1}{r^2} + \frac{d}{\lambda^2 R}$$

$$S \sim \frac{1}{r^4} + \frac{2d}{\lambda^2 r^3} + \frac{d^2}{\lambda^4 r^2} \sim 1, \quad \frac{2dr}{\lambda^2}, \frac{d^2 r^2}{\lambda^4}$$

if $d \ll \lambda$ still keep last term, if $r \gg \lambda$

Electric dipole radiation

- small source $d \ll \lambda$ $\beta \ll 1$ $k \approx 1$

Add up fields from each charge

$$(\vec{E}_a)_x = \frac{q_x}{c^2} \left[\frac{\hat{R}_x \times (\hat{R}_x \times \vec{a}_x)}{R_x} \right]$$

with one source charge \rightarrow Larmor formula.

add up sources.

if we ignore retardation w/in source (except for \vec{a}_x)
 $R \approx r$ since $r \gg d$

$$\rightarrow \vec{E}_{rad} = \sum_x (\vec{E}_a)_x = \frac{1}{c^2 r} \hat{n} \times (\hat{n} \times [\sum_x q_x \vec{a}_x])$$

express in terms of dipole moment:

$$\vec{p} = \sum_x q_x \vec{r}'_x \quad (\text{prime w/in source})$$

$$\ddot{\vec{p}} = \sum_x q_x \vec{a}_x$$

$$\rightarrow \vec{E}_{rad} = \frac{1}{c^2 r} \hat{n} \times (\hat{n} \times [\ddot{\vec{p}}])$$

we can use Larmor formula:

$$\frac{dP}{d\Omega} = \frac{e^2 [a^2]}{4\pi c^3} \sin^2 \theta \rightarrow \frac{[\ddot{p}^2]}{4\pi c^3} \sin^2 \theta$$

Time dependence $p(t) \sim p_0 e^{-i\omega t}$ or $p_0 \cos \omega t$
 $\rightarrow \langle \ddot{p}^2 \rangle = p_0^2 \omega^4 \langle \cos^2 \omega t \rangle = \frac{1}{2} p_0^2 \omega^4$

$$\langle \frac{dP}{d\Omega} \rangle = \frac{p_0^2 \omega^4}{8\pi c^3} \sin^2 \theta \quad \langle P \rangle = \frac{p_0^2 \omega^4}{3c^3} !$$

Example: radiation damping

electron bound in SHO potential. $F = \alpha X$ Hooke's law
 $t=0$ stretch, release.

$$m_e a(t) = \alpha x(t)$$

$$\ddot{x} = \frac{\alpha}{m_e} x \rightarrow x = X_0 \cos(\omega_0 t) \quad \omega_0 = \sqrt{\alpha/m_e}$$

$$\therefore \text{no damping, } a(t) = \frac{\alpha}{m_e} X_0 \cos(\omega_0 t) = \omega_0^2 X_0 \cos(\omega_0 t)$$

acceleration \rightarrow radiation, energy loss.

$$P = \frac{2}{3} \frac{e^2}{c^3} a^2(t) = \frac{2}{3} \frac{e^2}{c^3} \omega_0^4 X_0^2 \cos^2(\omega_0 t)$$

more power emitted at turning points

$$\text{time avg: (cycle avg)} \quad \langle \cos^2 \omega_0 t \rangle = \frac{1}{2}$$

$$\langle P \rangle = \frac{1}{3} \frac{(e X_0)^2 \omega_0^4}{c^3}$$



$$e X_0 = p = \text{dipole moment.} \quad \langle P \rangle = \frac{p^2 \omega_0^4}{3 c^3}$$

if electron is driven (say by EM wave) \rightarrow Rayleigh scattering.

for our problem, electron gradually loses energy

anticipate result:



$$x(t) = X_0(t) \cos(\omega_0 t)$$

\hookrightarrow envelope.

This is a tough, nonlinear problem:

$$-P = \frac{d}{dt} (E_{\text{rot}}) \Rightarrow \frac{d}{dt} \left(\frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2 \right) = - \frac{2}{3} \frac{e^2 (\ddot{x})^2}{c^3}$$

w/o damping, $E_{\text{tot}} = \frac{1}{2} m \omega_0^2 x_0^2 \sin^2 \omega_0 t$
 $+ \frac{1}{2} m \omega_0^2 x_0^2 \cos^2 \omega_0 t$
 $= \frac{1}{2} m \omega_0^2 x_0^2 = \text{constant.}$

\therefore slow decrease in energy \rightarrow slow decrease in amplit. x_0

write $\frac{d}{dt} \left(\frac{1}{2} m \omega_0^2 x_0^2(t) \right) = - \langle P \rangle =$
 $= - \frac{1}{3} \frac{e^2 \omega_0^4}{c^3} x_0^2(t)$

or write as diff eq. for E_{tot} : $x_0^2 = \frac{2 E_{\text{tot}}}{m \omega_0^2}$
 $\frac{d}{dt} E_{\text{tot}} = - \frac{1}{3} \frac{e^2 \omega_0^4}{c^3} \cdot \frac{2 E_{\text{tot}}}{m \omega_0^2}$
 $= - \frac{2}{3} \frac{e^2}{m c^2} \frac{\omega_0^2}{c} E_{\text{tot}}$

$\frac{e^2}{m c^2} = r_e$ classical electron radius $\sim 2.8 \times 10^{-15} \text{ m}$

$\dot{E}_{\text{tot}} = - \gamma_R E_{\text{tot}}$ $\gamma_R = \frac{2}{3} r_e \omega_0^2 / c$

$E_{\text{tot}} = E_0 e^{-\gamma_R t}$ radiation damping.

$\gamma_R = \frac{2}{3} r_e \frac{4\pi^2 c}{\lambda^2}$

At $\lambda = 500 \text{ nm}$ $1/\gamma_R = 10 \text{ ns}$ \approx natural decay time in atoms.

Can think of damping as force on electron by field.

\rightarrow "radiation reaction" very tough problem theoretically.
in antennas \rightarrow radiation resistance.