Scattering and dispersion theory.

main concept:

- incident wave $E(\vec{r},t) = \hat{\varepsilon} E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$
- drives oscillations in electrons in medium:

  e.g. single electron (low freq) → Thomson scatt.

  molecule → Rayleigh (high freq) → Compton scatt., moment. xfer.
- classical eqn. motion for electron

  $\vec{F} = q\vec{E} = m\ddot{\vec{r}}$

  > if electron is bound: add restoring force $K\vec{r}$

  or $m\omega_0^2\vec{r}$

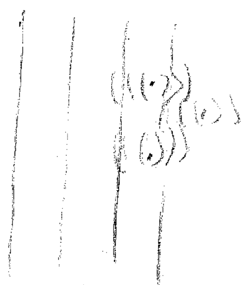
  > add damping $\gamma\dot{\vec{r}}$ radiative

  $m\gamma\dot{\vec{r}}$ collisional

- solution $\vec{r}(t)$ → dipole moment $\vec{p}(t) \sim q\vec{r}(t)$

  acceleration $\ddot{\vec{r}}$ → dipole radiation.

  - phase shift depends on $\omega - \omega_0$
- add scattered fields to incident wave



more dense material, smooth distrib.

→ coherent sum in forward direc.

resulting wave prop. at $v_{ph} \neq c$

→ index of refraction.

Thomson scattering: free electron, elastic (no Δp for e^-

$$\ddot{\vec{p}}(t) = q\ddot{r}(t) = \frac{q^2\vec{E}(t)}{m}$$

radiated power

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \underbrace{\frac{\langle [\ddot{p}]^2 \rangle}{4\pi c^3} \sin^2\theta}_{\text{Larmor}} = \frac{1}{2}\frac{q^4 E_0^2}{m^2} \cdot \frac{\sin^2\theta}{4\pi c^2}$$

$q = e$
$m = m_e$

$$= \left(\frac{e^2}{m_e c^2}\right)^2 \cdot \frac{c}{8\pi} E_0^2 \cdot \sin^2\theta$$

$$= r_e^2 I_0 \sin^2\theta$$

$\hookrightarrow$ classical el. radius in SI $\frac{e^2}{4\pi\varepsilon_0 m_e c^2}$

diff. cross-section

$$\frac{d\sigma}{d\Omega} \equiv \frac{\langle dP/d\Omega \rangle}{I} = r_e^2 \sin^2\theta$$

total cross-sect.:

$$\sigma = 2\pi \int_0^\pi r_e^2 \sin^2\theta \sin\theta \, d\theta$$

$$= \frac{8}{3} \cdot \pi r_e^2$$

for unpolarized light, avg. over $\vec{E}$ directions.

Can use Thomson scatt. $\rightarrow$ diagnostic of T_e in plasmas
- Doppler broadening of incident bandwidth.

Refractive index of gases:

- transition from micro- to macroscopic description.

$\vec{D} = \varepsilon \vec{E}$ in a linear medium.

$\varepsilon \equiv n^2$ comes from medium response.

Applied field induces a polarization in the medium:

$\vec{P} = \sum_\alpha N_\alpha \vec{p}_\alpha$

α = species index

N_α = number density

(book: $N f_\alpha$)

$\vec{D} \equiv \vec{E} + 4\pi \vec{P}$

explicitly shows how medium responds

if $\vec{P} \propto \vec{E}$ → linear, isotropic

→ $\vec{P} \equiv \chi_e \vec{E}$

$\vec{D} = \underbrace{(1 + 4\pi \chi_e)}_{\varepsilon} \vec{E}$

absorption → χ_e is complex.

if $\vec{P} \not\parallel \vec{E}$, $\vec{D}$ is redirected.

→ birefringence

$\varepsilon \to \overleftrightarrow{\varepsilon}$ tensor

if response is nonlinear, e.g. anharmonic osc.

write $P = \chi_e^{(1)} E + \frac{1}{2!}\chi_e^{(2)} E^2 + \frac{1}{3!}\chi_e^{(3)} E^3 \ldots$

→ nonlinear optics.

e.g. $P_{NL} = \frac{1}{2}\chi_e^{(2)} (E_0 \cos \omega t)^2$ oscillates at 2ω

→ SHG

Our procedure:

calculate $r'(t) \to p \to \vec{P} = \chi_e \vec{E} \to \varepsilon = 1 + 4\pi\chi_e$

Damped-driven oscillator:

let $\vec{E} = \hat{x} E_0 e^{-i\omega t}$ $\rightarrow$ $\vec{r}' = \hat{x}\, x(t)$ complex, but take real pt.

$$m\ddot{x} = -eE(t) - Kx - \ell\dot{x}$$

for no driving or damping, $x \sim \cos\omega_0 t$ $\quad \omega_0^2 = \frac{K}{m}$

$\rightarrow K = m\omega_0^2$ $\quad \omega_0 =$ resonance freq.

for convenience, express damping in terms of a frequency

$\beta \equiv \ell/2m$

$$\rightarrow \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = -\frac{e}{m} E_0 e^{-i\omega t} \qquad \omega \neq \omega_0 \text{ in general}$$

$x(t) = x_0 e^{-i\omega t}$ must respond at driving freq.

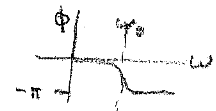
$$-\omega^2 x_0 - 2i\beta\omega x_0 + \omega_0^2 x_0 = -\frac{e}{m} E_0$$

$$p_0 = -e x_0 = \frac{(e^2/m)}{(\omega_0^2 - \omega^2) - 2i\beta\omega} E_0$$

note linear response. from $U(x) = \frac{1}{2} K x^2$

nonlinear: $U(x) = \frac{1}{2}Kx^2 + K^{(2)} x^3 + \ldots$

phase of x_0: $\omega \ll \omega_0$ in phase, $\omega \gg \omega_0$ out of phase



induced polarization:

$$\vec{P} = \underbrace{\left(\sum_\alpha \frac{N_\alpha e^2/m}{(\omega_\alpha^2 - \omega^2) - 2i\beta_\alpha\omega} \right)}_{\tilde{\chi}_e \text{ complex}} \vec{E}$$

$\tilde{\varepsilon} = 1 + 4\pi\tilde{\chi}_e$ $\qquad$ low density: $\tilde{n} \equiv \sqrt{\tilde{\varepsilon}} \approx 1 + 2\pi\tilde{\chi}_e$

refractive index is complex

$\tilde{n} = n_R + i n_I$

$n_R = 1 + 2\pi \, \mathrm{Re}(\tilde{\chi}_e)$ determines phase, group vel.

$n_I = 2\pi \, \mathrm{Im}(\tilde{\chi}_e)$ → absorption

$$\tilde{n}^2 = 1 + \sum_\alpha \frac{4\pi N_\alpha e^2/m}{\omega_\alpha^2 - \omega^2 - 2i\beta_\alpha \omega}$$

put into standard form

- $r_e = e^2/m_e c^2$
- introduce extra dimensionless factor:

  f_α = "oscillator strength" ~ 1 if strong transition

  quantum calc includes atomic details

$$\tilde{\varepsilon} = \tilde{n}^2 = 1 + \sum_\alpha \frac{4\pi r_e c^2 N_\alpha f_\alpha}{\omega_\alpha^2 - \omega^2 - 2i\beta_\alpha \omega}$$

Far from resonances $\tilde{\varepsilon}, \tilde{n}$ → real,

- no absorption, but index is determined by resonances

in gas, ω_0 in UV visible $\omega \ll \omega_0$

$N_a \sim 3\times 10^{19}/cm^3$ ω below resonance

→ $n-1 \approx +9\times 10^{-4}$

For $\omega \gg \omega_0$ (x-ray)

$n-1$ is negative ω above resonance.

Empirical forms away from resonance:

Cauchy, Sellmeier: based on fits to measurements near resonance: $\omega - \omega_0 \ll \omega_0$

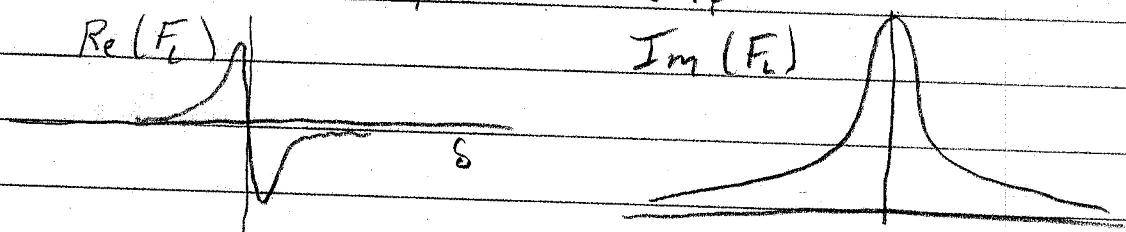
$$\omega_0^2 - \omega^2 - 2i\beta\omega = (\omega_0+\omega)(\omega_0-\omega) - 2i\beta\omega$$
$$\approx 2\omega_0\left((\omega_0-\omega) - i\beta\right)$$

$$\tilde{\varepsilon} = \tilde{n}^2 = 1 + \sum_\alpha \frac{2\pi r_e c^2 N_\alpha f_\alpha}{\omega_\alpha} F_L(\omega)$$

$$F_L(\omega) = \frac{1}{\omega_\alpha - \omega - i\beta} = \text{complex}$$

let $\delta = \omega_\alpha - \omega$

$$F_L = \frac{\delta}{\delta^2+\beta^2} + i\frac{\beta}{\delta^2+\beta^2}$$



absorption lineshape

F_L is "natural" lineshape

broader wings than Gaussian

note width: FWHM $\doteq \frac{\beta^2}{\delta^2+\beta^2}$ norm to 1

at $\delta = \beta \rightarrow 50\%$

$\therefore$ FWHM $= 2\beta = \gamma$ or A

$1/A$ is spontaneous emission lifetime

line broadening mechanisms:

- Doppler: $\Delta\omega = \omega(1 \pm v/c)$ avg over Boltzmann.
  $\rightarrow$ Gaussian lineshape.
- strain: local fields, Stark shift.
- power, pressure.